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MATHEMATICAL MONOGRAPHS.

EDITED BY

MANSFIELD MERRIMAN AND ROBERT S. WOODWARD.

No. 6.

GRASSMANN'S  
SPACE ANALYSIS.

BY

EDWARD W. HYDE,  
ACTUARY OF THE COLUMBIA INSURANCE COMPANY.

FOURTH EDITION.

FIRST THOUSAND.

NEW YORK:

JOHN WILEY & SONS.

LONDON: CHAPMAN & HALL, LIMITED.

1906.

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**BY**  
**MANSFIELD MERRIMAN AND ROBERT S. WOODWARD**  
**UNDER THE TITLE**  
**HIGHER MATHEMATICS.**

**First Edition, September, 1896.**  
**Second Edition, January, 1898.**  
**Third Edition, August, 1900.**  
**Fourth Edition, January, 1906.**

**ROBERT DRUMMOND, PRINTER, NEW YORK.**

## EDITORS' PREFACE.

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THE volume called Higher Mathematics, the first edition of which was published in 1896, contained eleven chapters by eleven authors, each chapter being independent of the others, but all supposing the reader to have at least a mathematical training equivalent to that given in classical and engineering colleges. The publication of that volume is now discontinued and the chapters are issued in separate form. In these reissues it will generally be found that the monographs are enlarged by additional articles or appendices which either amplify the former presentation or record recent advances. This plan of publication has been arranged in order to meet the demand of teachers and the convenience of classes, but it is also thought that it may prove advantageous to readers in special lines of mathematical literature.

It is the intention of the publishers and editors to add other monographs to the series from time to time, if the call for the same seems to warrant it. Among the topics which are under consideration are those of elliptic functions, the theory of numbers, the group theory, the calculus of variations, and non-Euclidean geometry; possibly also monographs on branches of astronomy, mechanics, and mathematical physics may be included. It is the hope of the editors that this form of publication may tend to promote mathematical study and research over a wider field than that which the former volume has occupied.

December, 1905.

## AUTHOR'S PREFACE.

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THIS little book is an attempt to present simply and concisely the elementary principles of the "Extensive Analysis" as fully developed in the comprehensive treatises of Hermann Grassmann, restricting the treatment however to the geometry of two and three dimensional space. It is designed to set forth, as far as is possible in so brief a work, the remarkable adaptability and effectiveness of the methods used as applied to the various problems and operations of geometry and mechanics.

The ideas of direction and position appear to the writer to be as simple and fundamental as that of magnitude, and an algebra which deals directly with all three of these ideas should not be greatly more difficult than the ordinary one, which deals with magnitude only. The result of using such an algebra is an extraordinary gain in the brevity of operations and the expressiveness of formulas and equations.

Some of the terms belonging to this general subject are frequently employed in modern text-books on mechanics and physics, even when no use is made of the algebraic systems from which they are derived.

It is hoped that this little book may do something to interest students, and to help toward bringing in the time when the methods as well as the ideas of this calculus shall come into general use.

CINCINNATI, O., December, 1905.

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# GRASSMANN'S SPACE ANALYSIS.

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## INTRODUCTION.

The title adopted for this brief and elementary discussion of Grassmann's methods indicates at once its limitations; for his theory in its fullness treats of the linear relation

$$p = \sum x_k e_k, \quad (k=0, 1, 2, \dots n),$$

in which  $n$  may be any positive whole number,  $x_0, x_1 \dots$  any numbers whatever, and  $e_0, e_1 \dots$  units of any kind which are susceptible of being related to each other by such an equation as the above. Our treatment extends only to the case when  $n$  does not exceed three, and  $e_0, e_1 \dots$  are points, or point products.

Grassmann's first publication of his new system was made in 1844 in a book entitled "Die Lineale Ausdehnungslehre. Ein Neuer Zweig der Mathematik." His novel and fruitful ideas were however presented in a somewhat abstruse and unusual form, with the result, as the author himself states in the preface to the second edition issued in 1878, that scarcely any notice was taken of the book by Mathematicians.

He was finally convinced that it would be necessary to treat the subject in an entirely different manner in order to gain the attention of the mathematical world. Accordingly in 1862 he published "Die Ausdehnungslehre, vollständig und in strenger Form bearbeitet," in which the treatment is algebraic, and is developed from the equation given above.

Since that time his great work has been more fully appreciated, but not even yet, in the opinion of the writer, at its real value.

Hamilton first gained the ear of the English-speaking world for his Quaternion methods, and was fortunate in having some very zealous adherents and interpreters who made propaganda for them, and were inclined to undervalue work not originating in England.

It is hoped that the following brief presentation of Grassmann's Analysis will serve to interest some to the extent that they may be led to investigate his original works.

The writer has followed in the main the notation of Grassmann himself, the principal deviations being the omission of the brackets from geometric products, writing  $pq$  instead of  $[pq]$ , and a somewhat different treatment of the product  $p|q$ .

#### ART. 1. EXPLANATIONS AND DEFINITIONS.

The algebra with which the student is already familiar deals directly with only one quality of the various geometric and mechanical entities, such as lines, forces, etc., namely, with their magnitude. Such questions as How much? How far? How long? are answered by an algebraic operation or series of operations. Questions of direction and position are dealt with indirectly by means of systems of coordinates of various kinds. In this chapter an algebra\* will be developed which deals directly with the three qualities of geometric and mechanical quantities, viz., magnitude, position, and direction. A geometric quantity may possess one, two, or all three of these properties simultaneously; thus a straight line of given length has all three, while a point has only one.

The geometric quantities with which we are to be concerned are the point, the straight line, the plane, the vector, and the plane-vector.

When the word "line" is used by itself, a "straight line" will be always intended. A portion of a given straight line of definite length will be called a "sect"; though when the length

\* The algebra of this chapter is a particular case of the very general and comprehensive theory developed by Hermann Grassmann, and published by him in 1844 under the title "Die lineale Ausdehnungslehre, ein neuer Zweig der Mathematik." He published also a second treatise on the subject in 1862.

of the sect is a matter of indifference, the word line will frequently be used instead. Similarly, a definite area of a given plane will be called a "plane-sect."

If a point recede to infinity, it has no longer any significance as regards position, but still indicates a direction, since all lines passing through finite points, and also through this point at infinity, are parallel. Similarly, a line wholly at infinity fixes a plane direction, that is, all planes passing through finite points, and also through this line at infinity, are parallel. Thus a point and line at infinity are respectively equivalent to a line direction and a plane direction.

A quantity possessing magnitude only will be termed a "scalar" quantity. Such are the ordinary subjects of algebraic analysis,  $a$ ,  $x$ ,  $\sin \theta$ ,  $\log z$ , etc., and they may evidently be intrinsically either positive or negative.

The letter  $T$  prefixed to a letter denoting some geometric quantity will be used to designate its absolute or numerical magnitude, always positive. Thus, if  $L$  be a sect, and  $P$  a plane-sect, then  $TL$  is the length of  $L$ , and  $TP$  is the area of  $P$ . That portion of a geometric quantity whose magnitude is unity will be called its "unit," and will be indicated by prefixing the letter  $U$ ; thus  $UL = \text{unit of } L = \text{sect one unit long on line } L$ .\* Hence we have  $TL \cdot UL = L$ .

## ART. 2. SUM AND DIFFERENCE OF TWO POINTS.

In geometric addition and subtraction we shall use the ordinary symbols  $+$ ,  $-$ ,  $=$ , but with modified significance, as will appear in the development of the subject.

Every mathematical, or other, theory rests on certain fundamental assumptions, the justification for these assumptions

\* The word "scalar" and the use of the letters  $T$  and  $U$ , as above, were introduced by Hamilton in his Quaternions.  $T$  stands for tensor, i.e., stretcher, and  $TL$  is the factor that stretches  $UL$  into  $L$ . The notation  $|L|$  for absolute magnitude is not used, because the sign  $|$  has been appropriated by Grassmann to another use.

lying in the harmony and reasonableness of the resulting theory, and its accordance with the ascertained facts of nature.

Our first assumption, then, will be that the associative and commutative laws hold for geometric addition and subtraction, that is, whatever  $A, B, C$  may represent, we have

$$\begin{aligned} A + B + C &= (A + B) + C = A + (B + C) \\ &= A + C + B = (A + C) + B. \end{aligned}$$

We shall also assume that we always have  $A - A = 0$ , and that the same quantity may be added to or subtracted from both sides of an equation without affecting the equality.

Now let  $p_1, p_2$  be two points, and consider the equation

$$p_2 + p_1 - p_1 = p_2 + (p_1 - p_1) = p_2. \quad (1)$$

In this form we have an identity. Write it, however, in the form

$$p_2 - p_1 + p_1 = (p_2 - p_1) + p_1 = p_2, \quad (2)$$

and it appears that  $p_2 - p_1$  is an operator that changes  $p_1$  into  $p_2$  by being added to it. Conceive this change of  $p_1$  into  $p_2$  to take place along the straight line through  $p_1$  and  $p_2$ ; then the operation is that of moving a point through a definite length or distance in a definite direction, namely, from  $p_1$  to  $p_2$ . This operator has been called by Hamilton "a vector,"\* that is, a carrier, because it carries  $p_1$  rectilinearly to  $p_2$ . Grassmann gives to it the name *Strecke*, and some writers now use the word "stroke" in the same sense.

Again,  $p_2 - p_1$  is the difference of two points, and the only difference that can exist between them is that of position, i.e. a certain distance in a certain direction.

Hence we may regard  $p_2 - p_1$  as a directed length, and also as the operator which moves  $p_1$  over this length in this direction. Writing  $p_2 - p_1 = \epsilon$ , equation (2) becomes

$$p_1 + \epsilon = p_2. \quad (3)$$

\* See the first of Hamilton's Lectures on Quaternions, where a very full discussion of equation (2) will be found. Also Grassmann (1862), Art. 227.

Thus the sum of a point and a vector is a point distant from the first by the length of the vector and in its direction.

Since  $p_2 - p_1 = -(p_1 - p_2)$ , it appears that the negative of a vector is a vector of the same length in the opposite direction.

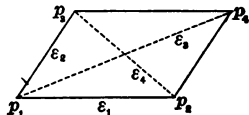
If  $p_2 - p_1 = 0$ , or  $p_2 = p_1$ ,  $p_2$  must coincide with  $p_1$  because there is now no difference between the two points.

The question arises as to what, if any, effect the operator  $p_2 - p_1$  should have on any other point  $p_3$ , that is, what is the value of the expression  $p_2 - p_1 + p_3$ ?

We will assume that it is some point  $p_4$ , so that we have  $p_2 - p_1 + p_3 = p_4$ ,

$$\text{or} \quad p_2 - p_1 = p_4 - p_3. \quad (4)$$

This implies that the transference from  $p_1$  to  $p_2$  is the same in amount and direction as that from  $p_3$  to  $p_4$ , that is, that  $p_1, p_2, p_3, p_4$  are the four corners of a parallelogram taken in order. Thus equal vectors have the same length and direction, and, conversely, vectors having the same length and direction are equal.



Note that parallel vectors of equal length are not necessarily equal, for their directions may be opposite.

Equation (4) may also be written

$$p_2 + p_3 = p_4 + p_1, \quad (5)$$

so that, whatever meaning may be assigned to the sum of two points, if we are to be consistent with assumptions already made, we must have the sum of either pair of opposite corner-points of a parallelogram equal to the sum of the other pair. The sum cannot therefore depend on the actual distances apart of the points forming the pairs, for the ratio of these two distances may be made as large or as small as we please.

If  $n$  be a scalar quantity,  $n\epsilon$  will denote that the operation  $\epsilon$  is to be performed  $n$  times on a point to which  $n\epsilon$  is added, that is, the point will be moved  $n$  times the length of  $\epsilon$ ; hence

$n\epsilon$  is a vector  $n$  times as long as  $\epsilon$ , and having the same or the opposite direction according to the sign of  $n$ .

In the figure above, let

$$p_2 - p_1 = \epsilon_1, \quad p_3 - p_1 = \epsilon_2, \quad p_4 - p_1 = \epsilon_3, \quad p_5 - p_1 = \epsilon_4.$$

Then

$$\epsilon_1 + \epsilon_2 = p_2 - p_1 + p_3 - p_1 = p_2 - p_1 + p_4 - p_2 = p_4 - p_1 = \epsilon_3, \quad (5)$$

since, by Eq. 4,  $p_3 - p_1 = p_4 - p_2$ .

$$\text{Also,} \quad \epsilon_3 - \epsilon_1 = p_4 - p_1 - p_2 = \epsilon_4. \quad (6)$$

Hence, if two vectors are drawn outwards from a point, and the parallelogram of which these are two adjacent sides is completed, then the two diagonals of this parallelogram will represent respectively the sum and difference of the two vectors, the sum being that diagonal which passes through the origin of the two vectors, and the difference that which passes through their extremities.\*

Again,  $p_2 - p_1 + p_3 - p_2 + p_4 - p_3 = 0 = \epsilon_1 + \epsilon_2 + (-\epsilon_3)$ ; hence the sum of three vectors represented by the sides of a triangle taken around in order is zero.

Similarly, if  $p_1, p_2, \dots, p_n$  be any  $n$  points whatever taken as corners of a closed polygon, we shall have

$$(p_2 - p_1) + (p_3 - p_2) + (p_4 - p_3) + \dots + (p_n - p_{n-1}) + (p_1 - p_n) = 0;$$

that is, the sum of vectors represented by the sides taken in order about the polygon is zero. By "taken in order" is not meant that any particular order of the points must be observed in forming the polygon; which is evidently unnecessary, but simply that, when the polygon is formed, the vectors will be the operators that will move a point from the starting position along the successive sides back to this position again, so that the final distance from the starting-point will be nothing.

### ART. 3. SUM OF TWO WEIGHTED POINTS.†

Consider the sum  $m_1 p_1 + m_2 p_2$ , in which  $m_1$  and  $m_2$  are scalars, that is, numbers, positive or negative, and  $p_1, p_2$  are points.

\* Grassmann (1844), § 15.

† Grassmann (1844), § 95, and (1862), Art. 227.

The scalars  $m_1$  and  $m_2$  will be regarded as values or weights assigned to the points  $p_1$  and  $p_2$ . When any weight is of unit value the figure 1 will be omitted, so that  $p$  means  $1p$ , and is called a unit point. Occasionally, however, a letter may be used to denote a point whose weight is not unity.

To assist his thinking, the reader may consider the weights initially as like or unlike parallel forces acting at the points.

In order to arrive at a meaning for the above expression we shall make two reasonable assumptions, which will prove to be consistent with those already made, viz., first, that the sum is a point, and second, that its weight is the sum of the weights of the two given points. Denoting this sum-point by  $\bar{p}$ , we write

$$m_1 p_1 + m_2 p_2 = (m_1 + m_2) \bar{p}. \quad (7)$$

Transposing, we have  $m_1(p_1 - \bar{p}) = m_2(\bar{p} - p_2)$ , or

$$\frac{p_1 - \bar{p}}{m_2} = \frac{\bar{p} - p_2}{m_1}. \quad (8)$$

Both members of (8) are vectors, and, being equal, they must, by Art. 4, be parallel. This requires that  $\bar{p}$  shall be collinear with  $p_1$  and  $p_2$ . Also, since  $p_1 - \bar{p}$  and  $\bar{p} - p_2$  are vectors whose lengths are respectively the distances from  $p_1$  to  $\bar{p}$  and from  $\bar{p}$  to  $p_2$ , it follows that these distances are in the ratio of  $m_2$  to  $m_1$ . Hence,  $\bar{p}$  is a point on the line  $p_1 p_2$  whose distances from  $p_1$  and  $p_2$  are inversely proportional to the weights of these points. We shall call  $\bar{p}$  the mean point of the two weighted points. If  $m_1$  and  $m_2$  are both positive, (8) shows that  $\bar{p}$  must lie between  $p_1$  and  $p_2$ ; but if one, say  $m_2$ , is negative, let  $m_2 = -m_2'$ . Thus

$$m_1(p_1 - \bar{p}) = m_2'(p_2 - \bar{p}), \quad (9)$$

and  $\bar{p}$  is on the same side of each point, that is, its direction from each point is the same. Also, since its distances from the two points are inversely as their weights,  $\bar{p}$  must be nearest the point whose weight is greatest.



Case when  $m_1 + m_2 = 0$ , or  $m_2 = -m_1$ .\*—With this condition equations (7) and (8) become

$$m_1 p_1 + m_2 p_2 = m_1 (p_1 - p_2) = 0 \cdot \bar{p}, \quad (10)$$

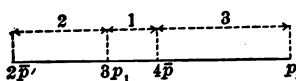
and

$$\bar{p} - p_1 = \bar{p} - p_2. \quad (11)$$

Thus  $\bar{p}$  is in the same direction from each point, that is, not between them, and yet is equidistant from them. This requires either that the two points shall coincide, that is,  $p_1 = p_2$ , which evidently satisfies (10) and (11); or else,  $p_1$  and  $p_2$  being different points, that  $\bar{p}$  shall be at an infinite distance. Thus the sum is in this case a point of zero weight at infinity.† Eq. (10) shows that a zero point at infinity is equivalent to a vector, or directed quantity, as stated in Art. 1. It has been shown in Art. 2 that  $p_2 = p_1$  is the condition that  $p_1$  and  $p_2$  coincide; let us consider the equality of weighted points in general, say  $m_1 p_1 = m_2 p_2$ . Hence, by (7), there is found  $m_1 p_1 - m_2 p_2 = (m_1 - m_2) \bar{p} = 0$ ; hence, since  $\bar{p}$  cannot be zero,  $m_1 - m_2 = 0$ , or  $m_1 = m_2$ ; and therefore  $m_1 (p_1 - p_2) = 0$ , or, since  $m_1 \geq 0$ ,  $p_1 - p_2 = 0$ , that is,  $p_1 = p_2$ . Therefore, if any two points are equal, their weights must be the same and their positions identical, that is, they are the same point.

Exercise 1.—To find the sum and difference of the two weighted points  $3p_1$  and  $p_2$ :

$$3p_1 + p_2 = 4\bar{p}, \quad 3p_1 - p_2 = 2\bar{p}',$$



and the mean points are as shown in the figure. The reciprocals of the distances of  $\bar{p}$ ,  $p_1$ , and  $\bar{p}'$  from  $p_2$ , viz.,  $\frac{1}{3}$ ,  $\frac{1}{4}$ ,  $\frac{1}{5}$ , are in arithmetical progression, hence the points form a harmonic range.

Exercise 2.—Given a circular disk with a circular disk of

\* Grassmann (1862), Art. 222.

† Compare the case of the resultant of unlike parallel forces of equal magnitude.

half its radius removed, as in the figure; to find the centroid of the remaining portion.

Take  $p_2$  at center of large circle,  $p_1$  at center of small circle, and  $p_3$  at the point of contact: then  $p_3 = \frac{1}{2} p_2 - p_1$ . The areas of the two circles are as 1 : 4: call them 1 and 4. Then it is as if there were a weight 4 at  $p_2$ , and a weight -1 at  $p_1$ ; hence  $\bar{p} = [4p_2 - 1p_1] \div 3 = (p_2 - p_1) \div 3$ .

PROB. 1. Show that  $p_1, p_2, p_3 = p_2 - \frac{1}{2} p_1$ , and  $\frac{1}{2} p_2 - p_1 = \frac{1}{2} p_3$ , the four points forming a harmonic range.

PROB. 2. An inscribed right-angled triangle is cut from a circular disk: show that the centroid of the remainder of the disk is at the point

$$\frac{3\pi - 2 \sin 2\theta}{3\pi - \sin 2\theta} \cdot \frac{p_2 - p_1 \sin 2\theta}{2},$$

if  $p_2$  is the center of the circle,  $p_1$  the opposite vertex of the triangle, and  $\theta$  one of its angles.

#### ART. 4. SUM OF ANY NUMBER OF POINTS.

As in the last article we assume the sum to be a point whose weight is equal to the sum of the weights of the given points: thus,

$$\sum w_i p_i = \bar{p} \sum w_i \quad (1)$$

Let  $e$  be some fixed point, and subtract  $e \sum w_i$  from both sides of (1): thus we have

$$\sum w_i p_i - e \sum w_i = \bar{p} \sum w_i - e \sum w_i \quad (2)$$

an equation which gives a simple construction for  $\bar{p}$ .

If  $\sum w_i = 1$  then  $\bar{p} = \sum w_i p_i$ , and

$$\sum w_i p_i = \bar{p} \sum w_i = \bar{p} \quad \text{if } \sum w_i = 1, \quad \text{and} \quad \bar{p} = \frac{\sum w_i p_i}{\sum w_i}.$$

so that the sum becomes the difference of two unit points, or a vector whose direction is parallel to the line joining  $p_1$  with the mean of all the other points of the system, and whose length is  $m_1$  times the distance between these points. Since any point of the system may be designated as  $p_1$ , it follows that the line joining any point of the system to the mean of all the others is parallel to any other such line. If  $\sum_1^{\infty} m p = 0$ , equation (14) shows that  $p$  is the mean of all the other points of the system, and, since any one of the points may be taken as  $p_1$ , any point of the system is the mean of all the others.

Let  $n = 3$  in (12) and (13); then

$$m_1 p_1 + m_2 p_2 + m_3 p_3 = (m_1 + m_2 + m_3) \bar{p}, \quad (15)$$

$$m_1(p_1 - e) + m_2(p_2 - e) + m_3(p_3 - e) = (m_1 + m_2 + m_3)(\bar{p} - e), \quad (16)$$

and  $\bar{p}$  is on the line joining the point  $m_1 p_1 + m_2 p_2$  with  $p_3$ , and therefore inside the triangle  $p_1 p_2 p_3$  if the  $m$ 's are all positive. If  $m_3$  be negative and numerically less than  $m_1 + m_2$ , then  $\bar{p}$  will have passed across the line  $p_1 p_2$  to the outside of the triangle. If  $m_1$  and  $m_2$  are negative and their sum numerically less than  $m_3$ , then  $\bar{p}$  will have passed outside the triangle through  $p_3$ , i.e., it will have crossed  $p_2 p_3$  and  $p_1 p_3$ . The point  $e$  must evidently always be in the plane  $p_1 p_2 p_3$ .

As a numerical example let  $m_1 = 3$ ,  $m_2 = 4$ ,  $m_3 = -5$ , so that (16) becomes

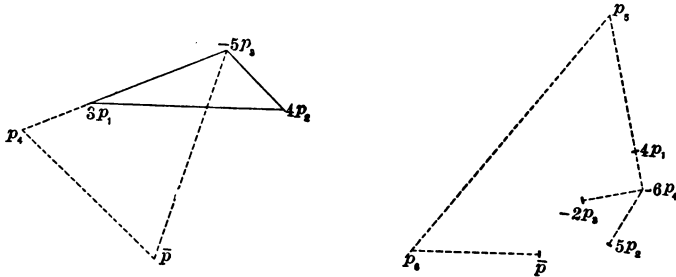
$$\bar{p} - e = \frac{3}{2}(p_1 - e) + 2(p_2 - e) - \frac{5}{2}(p_3 - e).$$

Now, since  $e$  may be any point whatever, put  $e = p_3$ ; then  $\bar{p} - p_3 = \frac{3}{2}(p_1 - p_3) + 2(p_2 - p_3)$ , and the construction is shown in the figure.  $p_1 - p_3 = \frac{3}{2}(p_1 - p_3)$ , and  $\bar{p} - p_3 = 2(p_2 - p_3)$ .

As another example take  $\bar{p} = 4p_1 + 5p_2 - 2p_3 - 6p_4$ , or, by (13), making  $e = p_4$ ,

$$\begin{aligned} \bar{p} - p_4 &= 4(p_1 - p_4) + 5(p_2 - p_4) - 2(p_3 - p_4) \\ &= p_1 - p_4 + p_2 - p_4 + \bar{p} - p_4. \end{aligned}$$

When any number of geometric quantities can be connected with each other by an equation of the form  $\sum m p = 0$ , in which the  $m$ 's are finite and different from zero, then they are said to be mutually dependent, that is, any one can be expressed in terms of the others. If no such relation can exist between the



quantities, they are independent. We obtain from what has preceded the following conditions:

That two points shall coincide,

$$m_1 p_1 + m_2 p_2 = 0. \quad (17)$$

That three points shall be collinear,

$$m_1 p_1 + m_2 p_2 + m_3 p_3 = 0. \quad (18)$$

That four points shall be coplanar,

$$m_1 p_1 + m_2 p_2 + m_3 p_3 + m_4 p_4 = 0. \quad (19)$$

It follows that three non-collinear points cannot be connected by an equation like (18) unless each coefficient is separately zero. Similarly four non-coplanar points cannot be connected by an equation like (19) unless each coefficient is separately zero.

The significance of these statements will be presently illustrated.

The following are corresponding equations of condition for vectors:

That two vectors shall be parallel,

$$n_1 \epsilon_1 + n_2 \epsilon_2 = 0. \quad (20)$$

That three vectors shall be parallel to one plane,

$$n_1\epsilon_1 + n_2\epsilon_2 + n_3\epsilon_3 = 0. \quad (21)$$

These conditions follow from the results of Art. 2, or from equations (17) and (18) by regarding the  $\epsilon$ 's as points at infinity. If in addition to (21) we have

$$n_1 + n_2 + n_3 = 0, \quad (22)$$

the extremities of the three vectors, if radiating from a point, will be collinear: for, let  $e_0 \dots e_3$  be four points so taken that  $e_1 - e_0 = \epsilon_1$ ,  $e_2 - e_0 = \epsilon_2$ ,  $e_3 - e_0 = \epsilon_3$ ; then (21) becomes

$$n_1(e_1 - e_0) + n_2(e_2 - e_0) + n_3(e_3 - e_0) = 0,$$

or by (22) 
$$n_1e_1 + n_2e_2 + n_3e_3 = 0,$$

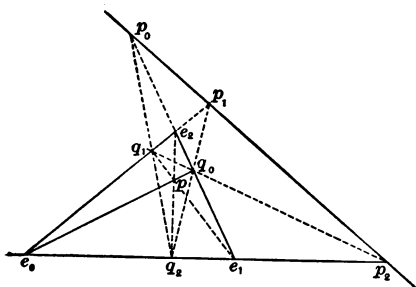
which by (18) requires  $e_1, e_2, e_3$  to be collinear.

It may be shown similarly that

$$\sum_1^4 n\epsilon = \sum_1^4 n = 0 \quad (23)$$

are the conditions that four vectors radiating from a point shall have their extremities coplanar.

Exercise 3.—Given a triangle  $e_0e_1e_2$  and a point  $p$  in its



plane;  $pe_0$  cuts  $e_1e_2$  in  $q_0$ ,  $pe_1$  cuts  $e_2e_0$  in  $q_1$ ,  $pe_2$  cuts  $e_0e_1$  in  $q_2$ ,  $q_1q_2$  cuts  $e_1e_2$  in  $p_0$ ,  $q_2q_0$  cuts  $e_2e_0$  in  $p_1$ , and  $q_0q_1$  cuts  $e_0e_1$  in  $p_2$ : to show that  $p_0, p_1$ , and  $p_2$  are collinear.

Let  $p = n_0e_0 + n_1e_1 + n_2e_2$ ; then  $q_0, q_1, q_2$  coincide respectively with  $n_1e_1 + n_2e_2$ ,

$n_2e_2 + n_0e_0$ , and  $n_0e_0 + n_1e_1$  because  $p$  lies on the line joining  $e_0$  with  $q_0$ , etc. Hence, if  $x_0, x_1, y_0, y_1$  are scalars,

$$p_2 = x_0e_0 + x_1e_1 = y_0(n_1e_1 + n_2e_2) + y_1(n_2e_2 + n_0e_0);$$

hence  $(x_0 - y_1n_0)e_0 + (x_1 - y_0n_1)e_1 - n_2(y_0 + y_1)e_2 = 0$ .

Now the  $e$ 's are not collinear, and yet are connected by a

relation of the form of equation (18); hence, as was there shown, each coefficient must be zero; accordingly

$$x_0 - y_1 n_0 = x_1 - y_0 n_1 = y_0 + y_1 = 0,$$

whence we find  $x_0 : x_1 = n_0 : -n_1$ .

hence  $(n_0 - n_1)p_2 = n_0 e_0 - n_1 e_1$ , and similarly

$$(n_1 - n_2)p_0 = n_1 e_1 - n_2 e_2, \quad (n_2 - n_0)p_1 = n_2 e_2 - n_0 e_0.$$

Adding, we have

$$(n_1 - n_2)p_0 + (n_2 - n_0)p_1 + (n_0 - n_1)p_2 = 0;$$

therefore, by (18),  $p_0, p_1, p_2$  are collinear.

Exercise 4.—Let  $p = \sum_0^2 n e \div \sum_0^2 n$  be any point in the plane of the triangle  $e_0 e_1 e_2$ : show that lines through the middle points of the sides  $e_1 e_2, e_2 e_0$ , and  $e_0 e_1$  of the triangle parallel to  $e_0 p, e_1 p$ , and  $e_2 p$  meet in a point

$$p' = [(n_1 + n_2)e_0 + (n_2 + n_0)e_1 + (n_0 + n_1)e_2] \div 2 \sum_0^2 n.$$

By the conditions the vector from the middle point of  $e_1 e_2$  to  $p'$  is a multiple of the vector  $e_0 - p$ ; hence

$$p' - \frac{1}{2}(e_1 + e_2) = x(e_0 - p) \quad \text{or}$$

$$p' = \frac{1}{2}(e_1 + e_2) + x(e_0 - p) = \frac{1}{2}(e_0 + e_1) + y(e_2 - p),$$

or, substituting value of  $p$ ,

$$p' = \frac{1}{2}(e_1 + e_2) + x(e_0 - \sum n e \div \sum n) = \frac{1}{2}(e_0 + e_1) + y(e_2 - \sum n e \div \sum n).$$

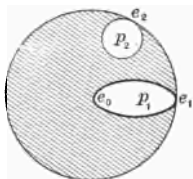
$$\begin{aligned} \text{hence} \quad & [(x - \frac{1}{2})\sum n + n_0(y - x)]e_0 + n_1(y - x)e_1 \\ & + [(\frac{1}{2} - y)\sum n + n_2(y - x)]e_2 = 0; \end{aligned}$$

therefore, as in the previous exercise, each coefficient must be zero, whence  $x = y = \frac{1}{2}$ , and substituting we find  $p'$  as above. It follows also that the distances of  $p'$  from the middle points of the sides are the halves of the distances of  $p$  from the opposite vertices.

Prob. 3. Show that  $\bar{e} = \frac{1}{3} \sum_0^2 e$  is collinear with  $p$  and  $p'$  of Exer-

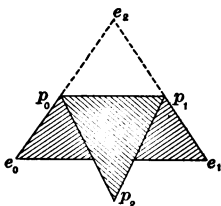
cise 4. Also that, by properly choosing  $p$ , it follows that  $\bar{e}$  is col-linear with the common point of the perpendiculars from the vertices on the opposite sides, and the common point of the perpendiculars to the sides at their middle points.

Prob. 4. Given two circles and an ellipse, as in the figure, with centers at  $e_0, p_2$ , and  $p_1$ . Radii of circles 4 and 1, axes of ellipse 2 and 4, small circle and ellipse touching large circle at  $e_2$  and  $e_1$  respectively,  $e_0, e_1, e_2$  an equilateral triangle: show that the centroid of the remainder of the large circle, after the small areas are removed, will be at



$\bar{p} = \frac{1}{13}(16e_0 - p_2 - 2p_1) = \frac{1}{13}(59e_0 - 4e_1 - 3e_2)$ .

$\bar{p} = \frac{1}{13}(16e_0 - p_2 - 2p_1) = \frac{1}{13}(59e_0 - 4e_1 - 3e_2)$ .



Prob. 5. If  $\frac{1}{8}$  of a sheet of tin in the shape of an isosceles triangle be folded over as in the figure, show that its centroid is given by  $3\bar{p} = \frac{1}{17}[35(e_0 + e_1) + 11e_2]$ .

Prob. 6. If a tetrahedron  $e_0e_1e_2e_3$  have a tetrahedron of  $\frac{1}{8}$  of its volume cut off by a plane parallel to  $e_0e_1e_2$ , and one of  $\frac{1}{4}$  of its volume cut off by a plane parallel to  $e_1e_2e_3$ , show that the centroid of the remaining solid is at

$$\bar{p} = \frac{1}{880}(227e_0 + 175e_2 + 239(e_1 + e_3)).$$

## ART. 5. REFERENCE SYSTEMS.

Let  $p$  be any unit point,  $e_0, e_1, e_2$  three fixed unit points, and  $w, x, y$  scalars; then, writing

$$p = we_0 + xe_1 + ye_2, \quad (24)$$

we must have also, because  $p$  is a unit point,

$$w + x + y = 1, \quad (25)$$

and  $p$  is the mean of the weighted points  $we_0, xe_1, ye_2$ . The point  $p$  may occupy any position whatever in the plane  $e_0e_1e_2$ ; for it is on the line joining  $we_0 + xe_1$  with  $e_2$ , and by varying  $y$  and  $w + x, \frac{w}{x}$  remaining constant,  $p$  may be moved along

this line from  $-\infty$  to  $+\infty$ ; while by varying the ratio  $\frac{w}{x}$  the point  $we_0 + xe_1$  may be moved from  $-\infty$  to  $+\infty$  along  $e_0e_1$ , and thus the first line will be rotated through 180 degrees, and  $p$  may thus be given any position whatever in the plane.

A system of unit points to which the positions of other points may be referred is called a reference system, and the triangle  $e_0e_1e_2$  is a reference triangle. For reasons that will appear later, the double area of this triangle will be taken as the unit of measurement of area for a point system in two-dimensional space.

Similarly, in solid space, taking a fourth point  $e_3$ , we write

$$p = we_0 + xe_1 + ye_2 + ze_3, \quad (26)$$

which implies also  $w + x + y + z = 1$ ; (27)

and  $p$  may be shown as above to be capable of occupying any position whatever in space by properly assigning the values of  $w, x, y, z$ ; so that  $e_0, \dots, e_3$  form a reference system for points in three-dimensional space. The tetrahedron  $e_0e_1e_2e_3$  is called the reference tetrahedron, and six times its volume will be taken as the unit of volume for a point system in three-dimensional space.

Eliminating  $w$  between (24) and (25), we have

$$p = e_0 + x(e_1 - e_0) + y(e_2 - e_0), \quad (28)$$

from which it may also be easily seen that  $p$  may be any point in the plane  $e_0e_1e_2$ . Writing  $p - e_0 = \rho$ ,  $e_1 - e_0 = \epsilon_1$ ,  $e_2 - e_0 = \epsilon_2$ ,

$$(28) \text{ becomes } \rho = x\epsilon_1 + y\epsilon_2, \quad (29)$$

and  $\epsilon_1, \epsilon_2$  form a plane reference system for vectors.

Similarly, from (26) and (27) we find

$$\rho = x\epsilon_1 + y\epsilon_2 + z\epsilon_3, \quad (30)$$

and  $\epsilon_1, \epsilon_2, \epsilon_3$  are a reference system for vectors in solid space, any vector whatever being expressible in terms of these three.

If, in equations (29) and (30), the reference vectors are of



unit length and mutually perpendicular, we have unit, normal reference systems, and in this case  $i, i, i$  will generally be used instead of  $\epsilon_1, \epsilon_2, \epsilon_3$ .

Exercise 5.—To change from one reference system to another, say from  $e_0, e_1, e_2$  to  $e'_0, e'_1, e'_2$ .

The new reference points must be connected with the old ones by equations such as

$$\begin{aligned} e_0 &= l_0 e'_0 + l_1 e'_1 + l_2 e'_2, & e_1 &= m_0 e'_0 + m_1 e'_1 + m_2 e'_2, \\ e_2 &= n_0 e'_0 + n_1 e'_1 + n_2 e'_2. \end{aligned}$$

Then any point  $p = x_0 e_0 + x_1 e_1 + x_2 e_2$  will be expressed in terms of the new reference points by substituting the values of  $e_0$ , etc., as given. If  $e'_0, e'_1, e'_2$  are given in terms of the old points,  $e_0, e_1, e_2$  may be found by elimination. Thus, if  $e'_0 = \sum l e$ ,  $e'_1 = \sum m e$ ,  $e'_2 = \sum n e$ , we have at once

$$\begin{vmatrix} l_0 & l_1 & l_2 \\ m_0 & m_1 & m_2 \\ n_0 & n_1 & n_2 \end{vmatrix} e_0 = \begin{vmatrix} e'_0 & l_1 & l_2 \\ e'_1 & m_1 & m_2 \\ e'_2 & n_1 & n_2 \end{vmatrix},$$

with similar values for  $e_1$  and  $e_2$ .

As a numerical example let the new reference triangle be formed by joining the middle points of the sides of the old one. Then  $e'_0 = \frac{1}{2}(e_1 + e_2)$ ,  $e'_1 = \frac{1}{2}(e_2 + e_0)$ ,  $e'_2 = \frac{1}{2}(e_0 + e_1)$ ; whence  $e_0 = -e'_0 + e'_1 + e'_2$ ,  $e_1 = e'_0 - e'_1 + e'_2$ ,  $e_2 = e'_0 + e'_1 - e'_2$ . Thus  $p = x_0 e_0 + x_1 e_1 + x_2 e_2$

$$= (-x_0 + x_1 + x_2)e'_0 + (x_0 - x_1 + x_2)e'_1 + (x_0 + x_1 - x_2)e'_2.$$

Exercise 6.—Three points being given in terms of the reference points  $e_0, e_1, e_2$ , find the condition that must hold between their weights when they are collinear.

Let  $p_0 = \sum_0^2 l e$ ,  $p_1 = \sum_0^2 m e$ ,  $p_2 = \sum_0^2 n e$ ; then,  $k, k_1, k_2$  being scalars, we must have for collinearity, by (18),

$$k_0 p_0 + k_1 p_1 + k_2 p_2 = 0,$$

that is,  $k_0 \sum l e + k \sum_1 m e + k \sum n e = 0$ ,

whence  $(k_0 l_0 + k_1 m_0 + k_2 n_0) e_0 + (k_0 l_1 + k_1 m_1 + k_2 n_1) e_1$   
 $+ (k_0 l_2 + k_1 m_2 + k_2 n_2) e_2 = 0$ ,

and, as  $e_0, e_1, e_2$  are not collinear, the coefficients must be zero, by Art. 4; hence

$$k_0 l_0 + k_1 m_0 + k_2 n_0 = k_0 l_1 + k_1 m_1 + k_2 n_1 = k_0 l_2 + k_1 m_2 + k_2 n_2 = 0,$$

and, by elimination of the  $k$ 's,

$$\begin{vmatrix} l_0 & m_0 & n_0 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0, \quad (31)$$

which is the required condition of collinearity.

Prob. 7. If  $p = 3e_0 - e_1 - e_2$ ,  $4e_0' = 3e_1 + e_2$ ,  $4e_1' = 3e_2 + e_0$ ,  $4e_2' = 3e_0 + e_1$ , show that  $7p = -19e_0' - 3e_1' + 29e_2'$ .

Prob. 8. Find the condition that four points  $\sum_0^3 k e, \sum_0^3 l e, \sum_0^3 m e, \sum_0^3 n e$  shall be coplanar. Ans.  $[k_0, l_1, m_2, n_3] = 0$ .

Prob. 9. If  $p = w e_0 + x e_1 + y e_2$ , and there exist between the scalars  $w, x, y$  a linear relation such as  $A w + B x + C y = 0$ ,  $A, B, C$  being scalar constants, show that  $p$  will always lie on a straight line which cuts the reference lines in  $A e_1 - B e_0, A e_2 - C e_0$ , and  $C e_1 - B e_2$ . Consider the special cases when  $A = B, B = C, C = A, A = B = C, A = 0, B = 0$ , and  $C = 0$ .

Prob. 10. If  $p = w e_0 + x e_1 + y e_2 + z e_3$ , and there exist also an equation  $A w + B x + C y + D z = 0$ , show that  $p$  will lie on a plane which cuts the edges of the reference tetrahedron in  $\frac{e_1}{B} - \frac{e_0}{A}, \frac{e_2}{C} - \frac{e_0}{A}$ , etc. . Also, if a second relation between the variables, such as  $A' w + B' x + C' y + D' z = 0$ , be given, then  $p$  lies on a line which pierces the faces of the reference tetrahedron in

$$\begin{vmatrix} e_0 & e_1 & e_2 \\ A & B & C \\ A' & B' & C' \end{vmatrix}, \quad \begin{vmatrix} e_1 & e_0 & e_2 \\ D & A & B \\ D' & A' & B' \end{vmatrix}, \quad \text{etc.}$$

## ART. 6. NATURE OF GEOMETRIC MULTIPLICATION.\*

The fundamental idea of geometric multiplication is, that a product of two or more factors is that which is determined by those factors.

Thus, two points determine a line passing through them, and also a length, viz., the shortest distance between them; hence  $p_1 p_2 = L$  is the sect † drawn from  $p_1$  to  $p_2$ , or generated by a point moving rectilinearly from  $p_1$  to  $p_2$ .

The student should note carefully the difference between  $p_1 p_2$  and  $p_2 - p_1$ ; they have the same length and direction, but the sect  $p_1 p_2$  is confined to the line through these two points, while the vector  $p_2 - p_1$  is not. The sect has position in addition to the direction and length possessed by the vector.

Again, in plane space, two sects determine a point, the intersection of the lines in which they lie, and also an area, as will appear later, so that  $L_1 L_2 = p$ , in which  $p$  is not in general a unit point. In solid space, however, two lines do not, in general, meet, and hence cannot fix a point; but two sects, in this case, determine a tetrahedron of which they are opposite edges.

It appears, therefore, that a product may have different interpretations in spaces of different dimensions. Hence we will consider separately products in plane space, or planimetric products, and those in solid space, or stereometric products.

Products of the kind here considered are termed "combinatory," because two or more factors combine to form a new quantity different from any one of them. This is the fundamental difference between this algebra and the linear associative algebras of Peirce, of which quaternions are a special case.

Before discussing in detail the various products that may arise, we will give a table which will serve as a sort of bird's-eye view of the subject.

\* Grassmann (1844), Chap. 2; (1862), Chap. 2.

† See Art. 1.

In this table and generally throughout the chapter we shall use  $p, p_1, p_2$ , etc., for points;  $\epsilon, \epsilon_1, \epsilon_2$ , etc., for vectors;  $L, L_1$ , etc., for sects, or lines;  $\eta, \eta_1$ , etc., for plane-vectors; and  $P, P_1$ , etc., for plane-sects, or planes. Also  $p, p_1$ , etc., as used in this table will not generally be unit points.

The products are arranged in two columns, so as to bring out the geometric principle of duality.

## PLANIMETRIC PRODUCTS.

$p_1 p_2 = L.$	$L_1 L_2 = p.$
$p_1 p_2 p_3 = \text{area (scalar).}$	$L_1 L_2 L_3 = (\text{area})^2 (\text{scalar}).$
$pL = \text{area (scalar).}$	$Lp = \text{area (scalar).}$
$p_1 \cdot L_1 L_2 = L.$	$L_1 \cdot p_1 p_2 = p.$
$p_1 p_2 \cdot p_3 p_4 = p.$	$L_1 L_2 \cdot L_3 L_4 = L.$
$p_1 p_2 \cdot p_3 p_4 \cdot p_5 p_6 = (\text{area})^2 (\text{scalar}).$	$L_1 L_2 \cdot L_3 L_4 \cdot L_5 L_6 = (\text{area})^4 (\text{scalar}).$
$\epsilon_1 \epsilon_2 = \text{area (scalar).}$	

## STEREOMETRIC PRODUCTS.

$p_1 p_2 = L.$	$P_1 P_2 = L.$
$p_1 p_2 p_3 = P.$	$P_1 P_2 P_3 = p.$
$p_1 p_2 p_3 p_4 = \text{volume (scalar).}$	$P_1 P_2 P_3 P_4 = (\text{volume})^3 (\text{scalar}).$
$pP = \text{volume (scalar).}$	$Pp = \text{volume (scalar).}$
$L_1 L_2 = \text{volume (scalar).}$	$L_1 L_2 = \text{volume (scalar).}$
$pL = Lp = P.$	$PL = LP = p.$
$p \cdot P_1 P_2 = P.$	$P \cdot p_1 p_2 = p.$
$p \cdot P_1 P_2 P_3 = L.$	$P \cdot p_1 p_2 p_3 = L.$
$L \cdot p_1 p_2 p_3 = p.$	$L \cdot P_1 P_2 P_3 = P.$
$\epsilon_1 \epsilon_2 = \eta.$	$\eta_1 \eta_2 = \epsilon.$
$\epsilon_1 \epsilon_2 \epsilon_3 = \text{volume (scalar).}$	$\eta_1 \eta_2 \eta_3 = (\text{volume})^2 (\text{scalar}).$
$\epsilon_1 \epsilon_2 \cdot \epsilon_3 \epsilon_4 = \epsilon.$	$\eta_1 \eta_2 \cdot \eta_3 \eta_4 = \eta.$

**Laws of Combinatory Multiplication.**—All combinatory products are assumed to be subject to the distributive law expressed by the equation

$$A(B + C) = AB + AC.$$

The planimetric product of three points or of three lines, and the stereometric product of three points or planes, or of four points or planes, are subject to the associative law. That is,

In Plane Space :

$$p_1 p_2 p_3 = p_1 p_3 p_2 = p_3 p_1 p_2; \quad L_1 L_2 L_3 = L_1 L_3 L_2 = L_3 L_1 L_2.$$

In Solid Space :

$$p_1 p_2 p_3 = p_1 p_3 p_2 = p_3 p_1 p_2; \quad P_1 P_2 P_3 = P_1 P_3 P_2 = P_3 P_1 P_2.$$

$$p_1 p_2 p_3 p_4 = p_1 p_3 p_2 p_4 = p_1 p_3 p_4 p_2;$$

$$P_1 P_2 P_3 P_4 = P_1 P_3 P_2 P_4 = P_3 P_1 P_2 P_4.$$

The commutative law of scalar algebra does not, in general, hold. Instead of this, in the products just given as being associative, a law prevails which may be expressed by the equation

$$AB = -BA,$$

from which it follows that the interchange of any two single factors of those products changes the sign of the product.\*

Since vectors are equivalent to points at  $\infty$ , the associative law holds for  $\epsilon_1 \epsilon_2 \epsilon_3$  and  $\eta_1 \eta_2 \eta_3$ .

## ART. 7. PLANIMETRIC PRODUCTS.

**Product of Two Points.**†—This has been fully defined in Art. 6, and it is evident from its nature as there given that

$$p_1 p_2 = -p_2 p_1. \quad (32)$$

If  $p_2 = p_1$ , this becomes  $p_1 p_1 = 0$ , which must evidently be true, since the sect is now of no length.

$$\text{Also,} \quad p_1(p_2 - p_1) = p_1 p_2 - p_1 p_1 = p_1 p_2. \quad (33)$$

\* Grassmann (1862), Chap. 3.      † Grassmann (1862). Arts. 245, 246, 247.

But  $p_2 - p_1$  is a vector, say,  $\epsilon$ ; hence

$$p_1 \epsilon = p_1 p_2; \quad (34)$$

or the product of a point and a vector is a sect having the direction and magnitude of the vector; or, again, multiplying a vector by a point fixes its position by making it pass through the point.

To find under what conditions  $pp'$  will be equal to  $p_1 p_2$ . Take any other point  $p_3$  in the plane space under consideration, and write  $p = x_1 p_1 + x_2 p_2 + x_3 p_3$ ,  $p' = y_1 p_1 + y_2 p_2 + y_3 p_3$ , with the conditions for unit points  $\sum x = \sum y = 0$ .

$$\text{Then } pp' = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} p_1 p_2 + \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} p_2 p_3 + \begin{vmatrix} x_3 & x_1 \\ y_3 & y_1 \end{vmatrix} p_3 p_1.$$

If this is to reduce to  $p_1 p_2$ , we must have the third condition  $x_2 y_3 - x_3 y_2 = x_3 y_1 - x_1 y_3 = 0$ , which requires that  $x_3 = y_3 = 0$ , unless the coefficient of  $p_1 p_2$  is to vanish also. Thus  $pp'$  must be in the same straight line with  $p_1 p_2$ . If, moreover, in addition  $x_1 y_3 - x_3 y_1 = 1$ , we shall have  $pp' = p_1 p_2$ . Hence  $pp'$  is equal to  $p_1 p_2$  when, and only when, the four points are collinear, and  $p'$  is distant from  $p$  by the same amount and in the same direction that  $p_2$  is from  $p_1$ .

Product of Three Points.—By Art. 6 the product is what is determined by the three points. In solid space they would fix a plane, but, as we are now confined to plane space, this is not the case. The points evidently fix either a triangle or a parallelogram of twice its area, and the product  $p_1 p_2 p_3$  will be taken as the area of this, or an equivalent, parallelogram.

This area is taken rather than that of the triangle, because it is what is generated by  $p_1 p_2$ , as it is moved parallel to its initial position till it passes through  $p_3$ .

We have  $p_1 p_2 p_3 = p_1 \cdot p_2 p_3 = -p_1 \cdot p_3 p_2 = -p_1 p_3 p_2$ , so that if we go around the triangle in the opposite sense the sign is changed. As this product possesses only the properties of magnitude and sign it is scalar.

$$\text{Write } p = \sum_1^3 x p, p' = \sum_1^3 y p, p'' = \sum_1^3 z p; \text{ then}$$

$$pp'p'' = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} p_1 p_2 p_3; \quad (35)$$

that is, any triple point product in plane space differs from any other only by a scalar factor.\*

$$\text{Finally, } p_1 p_2 p_3 = p_1 (p_2 - p_1) (p_3 - p_1) = p_1 \epsilon \epsilon', \quad (36)$$

if  $\epsilon = p_2 - p_1$  and  $\epsilon' = p_3 - p_1$ .

**Product of Two Vectors.**—Using the values of  $\epsilon$  and  $\epsilon'$  just given, we see that  $\epsilon$  and  $\epsilon'$  determine the same parallelogram that  $p_1$ ,  $p_2$ , and  $p_3$  do; hence the meaning of the product is the same in all respects in two-dimensional space.

We shall have  $\epsilon \epsilon' = -\epsilon' \epsilon$ , for

$$\epsilon \epsilon' = (p_2 - p_1)(p_3 - p_1) = -(p_3 - p_1)(p_2 - p_1) = -\epsilon' \epsilon;$$

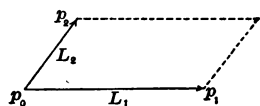
since we have shown that inverting the order changes the sign in a product of points. The result may be obtained also by regarding  $\epsilon$  and  $\epsilon'$  as points at infinity, or by consideration of a figure.

As we have seen that  $\epsilon \epsilon'$  has, in plane space, precisely the same meaning as  $p_1 p_2 p_3$ , we may write

$$\begin{aligned} p_1 p_2 p_3 &= p_1 \epsilon \epsilon' = \epsilon \epsilon' \\ &= (p_2 - p_1)(p_3 - p_1) = p_2 p_3 + p_3 p_1 + p_1 p_2. \end{aligned} \quad (37)$$

Thus the sum of three sects which form the sides of a triangle, all taken in the same sense as looked at from outside the triangle, is equal to the area of the triangle.

**Product of Two Sects.**—Any two sects in plane space,  $L_1$ ,  $L_2$ , determine a point, the intersection of the lines in which they lie, and



an area, that of a parallelogram as in the figure. Let  $p_0$  be the intersection, and take  $p_1$  and  $p_2$  so that  $L_1 = p_0 p_1$  and  $L_2 = p_0 p_2$ . The area

\* Grassmann (1862), Art. 255.

determined by  $L_1$  and  $L_2$  is then the same that we have given as the value of  $p_0 p_1 p_2$ . We write therefore

$$L_1 L_2 = p_0 p_1 \cdot p_0 p_2 = p_0 p_1 p_2 \cdot p_0. \quad (38)$$

The third member of (38) is not to be regarded as derived from the second by ordinary transposition and reassociation of the points, for the associative law does not hold for the four points taken together, since  $p_0 p_1 p_0 \cdot p_2 = 0$ . The third member simply results from the definition of  $L_1 L_2$ .<sup>\*</sup> It may be taken as a model form which will be found to apply to several other cases, for instance to (38) when points and lines are interchanged throughout. Thus, if  $p_1 = L_0 L_1$  and  $p_2 = L_0 L_2$ , we have

$$p_1 p_2 = L_0 L_1 \cdot L_0 L_2 = L_0 L_1 L_2 \cdot L_0. \quad (39)$$

For take  $p_1'$  and  $p_2'$  so that  $p_1 p_1' = L_1$  and  $p_2 p_2' = L_2$ ;  $p_1 p_2$  is evidently some multiple of  $L_0$ , say  $nL_0$ ; hence

$$\begin{aligned} p_1 p_2 &= nL_0 = \frac{1}{n^2} (p_1 p_2 \cdot p_1 p_1') \cdot (p_1 p_2 \cdot p_2 p_2') \\ &= \frac{1}{n^2} (p_1 p_2 p_1' \cdot p_1) \cdot (p_1 p_2 p_2' \cdot p_2), \text{ by (38),} \\ &= \frac{1}{n^2} \cdot p_1 p_2 p_1' \cdot p_1 p_2 p_2' \cdot p_1 p_2, \text{ because } p_1 p_2 p_1' \text{ and} \\ &\quad p_1 p_2 p_2' \text{ are scalar,} \\ &= \frac{1}{n} \cdot (p_1 p_2 \cdot p_1 p_1' \cdot p_2 p_2') \cdot L_0, \text{ by (38),} \\ &= L_0 L_1 L_2 \cdot L_0, \text{ which was to be proved.} \end{aligned}$$

**Product of Three Sects.**—The method has just been indicated, but we may also proceed thus: Let the lines be  $L_0, L_1, L_2$ , and let  $p_0, p_1, p_2$  be their common points. Take scalars  $n_0, n_1, n_2$  so that  $L_0 = n_0 p_1 p_2$ , etc., then

$$\begin{aligned} L_0 L_1 L_2 &= n_0 n_1 n_2 \cdot p_1 p_2 \cdot p_2 p_0 \cdot p_0 p_1 = -n_0 n_1 n_2 \cdot p_2 p_1 \cdot p_2 p_0 \cdot p_0 p_1 \\ &= -n_0 n_1 n_2 \cdot p_2 p_1 p_0 \cdot p_2 p_0 p_1 = n_0 n_1 n_2 (p_0 p_1 p_2)^2. \end{aligned} \quad (40)$$

<sup>\*</sup> Grassmann applies the terms "eingewandt" and "regressiv" to a product of this kind, the first term being used in the *Ausdehnungslehre* of 1844, and the second in that of 1862. See Chapter 3 of the first, and Chapter 3, Art. 94, of the second.



Product of a Point and Two Sects.—Let  $p$  be any point and let  $L_1$  and  $L_2$  be as in (38); then

$$pL_1L_2 = p \cdot p_0p_1 \cdot p_0p_2 = p \cdot p_0p_1p_2 \cdot p_0 = p_0\hat{p}_1p_2 \cdot pp_0. \quad (41)$$

It has been here assumed that  $pL_1L_2 = p \cdot L_1L_2$ . The product is not associative, for  $pL_1 \cdot L_2$  is the line  $L_2$  times the scalar  $pL_1$ , a different meaning from that assigned in (41). As a rule, to avoid ambiguity, the grouping of such products will be indicated by dots.

Product of Two Parallel Sects.—Let them be  $p_1\epsilon$  and  $n_2p_2\epsilon$ ; then, as in (38),

$$p_1\epsilon \cdot n_2p_2\epsilon = n \cdot p_1\epsilon \cdot p_2\epsilon = n \cdot \epsilon p_1 \cdot \epsilon p_2 = n \cdot \epsilon p_1p_2 \cdot \epsilon, \quad (42)$$

that is, a scalar times the common point at  $\infty$ .

Addition and Subtraction of Sects.—Let  $L_1$  and  $L_2$  be two sects,  $p_0$  their common point, and  $p_1$  and  $p_2$  so taken that  $L_1 = p_0p_1$ ,  $L_2 = p_0p_2$ ; then

$$L_1 + L_2 = p_0p_1 + p_0p_2 = p_0(p_1 + p_2) = 2p_0\bar{p}, \quad (43)$$

$\bar{p}$  being the mean of  $p_1$  and  $p_2$ ; hence the sum is that diagonal of the parallelogram which passes through  $p_0$ . Also

$$L_1 - L_2 = p_0(p_1 - p_2), \quad (44)$$

so that the difference of the two passes also through  $p_0$  and is parallel to the other diagonal of the parallelogram determined by  $L_1$  and  $L_2$ .

If the two sects are parallel let them be  $n_1p_1\epsilon$  and  $n_2p_2\epsilon$ ; then

$$n_1p_1\epsilon + n_2p_2\epsilon = (n_1p_1 + n_2p_2)\epsilon = (n_1 + n_2)\bar{p}\epsilon, \quad (45)$$

so that the sum is a sect parallel to each of them, having a length equal to the sum of their lengths, and at distances from them inversely proportional to their lengths.

If  $n_2 = -n_1$  the two sects are oppositely directed and of equal length, and the sum is

$$n_1(p_1\epsilon - p_2\epsilon) = n_1(p_1 - p_2)\epsilon, \quad (46)$$

which, being the product of two vectors, is a scalar area.

Consider next  $n$  sects  $p_1\epsilon_1, p_2\epsilon_2, \dots, p_n\epsilon_n$ , and let  $e_0$  be some arbitrarily chosen point; then

$$\sum_1^n p\epsilon \equiv e_0 \sum_1^n \epsilon - e_0 \sum_1^n \epsilon + \sum_1^n p\epsilon \equiv e_0 \sum_1^n \epsilon + \sum_1^n (p - e_0)\epsilon. \quad (47)$$

The second term of the third member of this equation, being a sum of double vector products, that is, a sum of areas, is itself an area, and is equal to the product of any two non-parallel vectors of suitable lengths. Therefore,  $\alpha$  and  $\beta$  being such vectors, write  $\sum \epsilon = \alpha$  and  $\sum (p - e_0)\epsilon = \alpha\beta$ . Hence (47) become

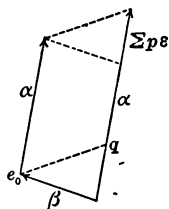
$$\sum p\epsilon = e_0\alpha + \alpha\beta = (e_0 - \beta)\alpha. \quad (48)$$

Let  $q$  be some point on the line  $\sum p\epsilon$ ; then

$$q\sum p\epsilon = 0 = qe_0\alpha + q\alpha\beta = qe_0\alpha + \alpha\beta,$$

by (37), hence  $qe_0\alpha = -\alpha\beta = \beta\alpha$ .

The figure presents the geometrical meaning of the equation, and hence it appears that  $q\alpha (= \sum p\epsilon)$  is at a perpendicular distance from  $e_0$  of



$$\frac{\alpha\beta}{T\alpha} = \frac{\sum (p - e_0)\epsilon}{T\sum \epsilon}. \quad (49)$$

It is easily seen that a sect possesses the exact geometrical properties of a force, namely, magnitude, direction, and position, and the discussion of the summation of sects which has just been given corresponds completely to the discussion of the resultant of a system of forces in a plane. In this algebra, then, the resultant of any system of forces is simply their sum, and this will be found hereafter to be equally true in three-dimensional space. The expression in (46) corresponds to a couple, as does also the  $\sum (p - e_0)\epsilon$  of (47); and this equation proves the proposition that any system of forces in a plane is equivalent to a single force acting at an arbitrary point,  $e_0$ , and a couple. Equation (49) gives the distance of the resultant from this arbitrary point.

Exercise 7.—To find  $x, y, z$  from the scalar equations

$$a_1x + b_1y + c_1z = d_1, \quad a_2x + b_2y + c_2z = d_2, \quad a_3x + b_3y + c_3z = d_3.$$

Multiply the equations by  $p_1$ ,  $p_2$ , and  $p_3$  respectively, and add; hence

$$x \sum_1^3 ap + y \sum_1^3 bp + z \sum_1^3 cp = \sum_1^3 dp.$$

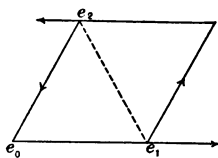
Now  $\sum ap$ ,  $\sum bp$ , etc., are points: multiply the equation just written by  $\sum ap$ ,  $\sum bp$ ; thus

$$z \sum ap \cdot \sum bp \sum cp = \sum ap \cdot \sum bp \cdot \sum dp,$$

because  $\sum ap \cdot \sum ap = 0$ , etc.; therefore

$$z = \sum ap \cdot \sum bp \cdot \sum dp \div \sum ap \cdot \sum bp \sum cp = [a_1, b_2, d_3] \div [a_1, b_2, c_3],$$

a very simple proof of the determinant solution. Of course  $x$  and  $y$  will be found by multiplying by the other pairs of points.



Exercise 8.—Forces are represented by given multiples of the sides of a parallelogram; determine their resultant.

Let the parallelogram be double the triangle  $e_0e_1e_2$ , and the forces

$$\begin{aligned} k_0e_0e_1 + k_1e_1(e_2 - e_0) + k_2e_2(e_0 - e_1) + k_3e_3e_0 &= \sum pe \\ &= (k_0 + k_1)e_0e_1 + (k_1 + k_2)e_1e_2 + (k_2 + k_3)e_2e_0. \end{aligned}$$

Multiply by  $e_0e_1$  to find where the resultant cuts this line, then

$$(k_1 + k_2)e_0e_1 \cdot e_1e_2 + (k_2 + k_3)e_0e_1 \cdot e_2e_0 = e_0e_1e_2 \cdot [(k_1 + k_2)e_1 - (k_2 + k_3)e_0],$$

or  $e_0e_1$  cuts the resultant at the point

$$[(k_1 + k_2)e_1 - (k_2 + k_3)e_0] \div (k_1 - k_2).$$

Similarly the resultant cuts the other sides of the reference triangle at  $[(k_2 + k_3)e_2 - (k_0 + k_1)e_1] \div (k_2 + k_3 - k_0 - k_1)$  and at  $[(k_0 + k_1)e_0 - (k_1 + k_2)e_2] \div (k_0 - k_2)$ .

Suppose  $k_0 = k_1 = k_2 = k_3$ ; then each of the three points just found recedes to infinity; but in this case  $\sum pe$  reduces to  $2k_0(e_0e_1 + e_1e_2 + e_2e_0) = 2k_0(e_1 - e_0)(e_2 - e_0)$ , and the system is equivalent to a couple.

Prob. 11. Construct the resultant of Exercise 8 when  $k_0 = 1$ ,  $k_1 = 2$ ,  $k_2 = 3$ ,  $k_3 = 4$ ; when  $k_0 = 1$ ,  $k_1 = -2$ ,  $k_2 = 3$ ,  $k_3 = -4$ ; when  $k_0 = 3$ ,  $k_1 = k_2 = 2$ ,  $k_3 = 1$ ; and when  $k_1 = k_2 = 1$ ,  $k_0 = k_3 = -2$ .

Prob. 12. There are given  $n$  points  $p_1 \dots p_n$ ; to find a point  $e$  such that forces represented by the sects  $ep_1, ep_2$ , etc., shall be in equilibrium. (The equation of equilibrium is  $\sum ep \equiv e \sum p \equiv \frac{1}{n} e \bar{p} = 0$ . Hence  $e$  coincides with the mean point of the  $p$ 's.)

Prob. 13. If a harmonic range  $e_1, p, e_2, p'$  be given, together with some point  $e_0$  not collinear with these points, show that

$$e_0 e_1 p \cdot e_0 e_2 p' = -e_0 p e_2 \cdot e_0 p' e_1.$$

(Let  $p = m_1 e_1 + m_2 e_2$  and  $p' = m_1 e_1 - m_2 e_2$ , as in Exercise 2 of Art. 3.)

Prob. 14. Show that the relation of Prob. 13 holds for any four points whatever taken respectively on the four lines  $e_0 e_1, e_0 p, e_0 e_2, e_0 p'$ . If the four points are all at the same distance from  $e_0$ , show that the areas  $e_0 e_1 p$ , etc., become proportional to the sines of the angles between  $e_0 e_1$  and  $e_0 p$ , etc.

## ART. 8. THE COMPLEMENT.\*

Taking point reference systems, or unit normal vector reference systems, as in Art. 5, the product of the reference units taken in order being in any case unity, the complement of any reference unit is the product of all the others so taken that the unit times its complement is unity.

To find the complements of quantities other than reference units the following properties are assumed:

(a) The complement of a product is equal to the product of the complements of its factors.

(b) The complement of a sum is equal to the sum of the complements of the terms added together.

(c) The complement of a scalar quantity is the scalar itself.

Considering now the point system in plane space  $e_0, e_1, e_2$ , with the constant condition  $e_0 e_1 e_2 = 1$ , the sides of the reference triangle taken in order are the complements of the opposite vertices, and vice versa.

The complement of a quantity is indicated by a vertical line, as  $|p$ , read, complement of  $p$ .

\* See *Ausdehnungslehre* of 1862, Art. 89.

Thus

$$\begin{aligned} |e_0 &= e_1 e_2, & |e_1 e_2 &= |(|e_0) = e_0, \\ |e_1 &= e_2 e_0, & |e_2 e_0 &= |(|e_1) = e_1, \\ |e_2 &= e_0 e_1, & |e_0 e_1 &= |(|e_2) = e_2. \end{aligned}$$

For  $e_0 | e_0 = e_0 e_1 e_2 = 1$ , which agrees with the definition ;

$|e_1 e_2 = |e_1 \cdot |e_2 = e_2 e_0 \cdot e_0 e_1 = -e_0 e_2 \cdot e_0 e_1 = -e_0 e_2 e_1 \cdot e_0 = e_0$ , by (a) and (38) ;

$|e_0 e_1 e_2 = |e_0 \cdot |e_1 \cdot |e_2 = e_1 e_2 \cdot e_2 e_0 \cdot e_0 e_1 = (e_0 e_1 e_2)^2 = 1 = e_0 e_1 e_2$ , which agrees with (c) ;  $e_0 | e_1 = e_0 e_2 e_0 = 0 = e_0 | e_2 = e_1 | e_2$ .

Next take any point  $p_1 = \sum_0^2 l e$ , and we have, by (b),

$$|p_1 = \sum_0^2 l |e = l_0 e_1 e_2 + l_1 e_2 e_0 + l_2 e_0 e_1 = l_0 l_1 l_2 \left( \frac{e_1}{l_1} - \frac{e_0}{l_0} \right) \left( \frac{e_2}{l_2} - \frac{e_0}{l_0} \right) = L_1. \quad (50)$$

Thus the complement of a point is a line,\* which may be easily constructed by the fourth member of (50), which expresses this line as the product of the points in which it cuts the sides  $e_0 e_1$  and  $e_0 e_2$  of the reference triangle. Comparing this equation with Ex. 3 in Art. 4, it appears that  $|p_1$  above is related to the point  $\sum_0^2 \frac{e}{l}$  as the line  $p_0 p_2$  of Ex. 3 is to the point  $\sum ne$ . Hence  $|p_1$  may be found by constructing this line corresponding to  $\sum_0^2 \frac{e}{l}$  as shown in the figure of Ex. 3, Art. 4.

Again, the line  $|p_1$  may be shown to be the anti-polar of  $p$  with respect to an ellipse of such dimensions, and so placed upon  $e_0 e_1 e_2$  that, with reference to it, each side of the reference triangle is the anti-polar of the opposite vertex.\* From this it appears that complementary relations are polar reciprocal relations. Take any point  $p_2 = \sum_0^2 m e$ , and we have

$$\begin{aligned} p_1 | p_2 &= (l_0 e_0 + l_1 e_1 + l_2 e_2)(m_0 e_1 e_2 + m_1 e_2 e_0 + m_2 e_0 e_1) \\ &= \sum_0^2 l m = \sum m e \cdot \sum l |e = p_2 | p_1, \end{aligned} \quad (51)$$

\* See Hyde's Directional Calculus, Arts. 41-43 and 121-123.

so that this product is commutative about the complement sign, and scalar. This is true of all such products when the quantities on each side of the complement sign are of the same order in the reference units. Take for instance the product  $p_1 p_2 | p_3 p_4$ . This is scalar because  $| p_3 p_4$  is a point, so that the whole quantity is equivalent to a triple-point product; and we have  $p_1 p_2 | p_3 p_4 = | p_3 p_4 \cdot p_1 p_2 = | (p_3 p_4 | p_1 p_2) = p_3 p_4 | p_1 p_2$ , by (a) and (c). If, however, such a quantity be taken as  $p_1 p_2 \cdot | p_3$ , it is neither scalar nor commutative about the sign  $|$ ; for,  $| p_3$  being a line, the product is that of two lines, that is, a point, and

$$p_1 p_2 \cdot | p_3 = - | p_3 \cdot p_1 p_2 = - | (p_3 \cdot p_1 p_2). \quad (52)$$

Such products as we have just been considering are called by Grassmann "inner products,"\* and he regards the sign  $|$  as a multiplication sign for this sort of product. Inasmuch, however, as these products do not differ in nature from those heretofore considered, it appears to the author to conduce to simplicity not to introduce a nomenclature which implies a new species of multiplication. For instance,  $p|q$  will be treated as the combinatory product of  $p$  into the complement of  $q$ , and not as a different kind of product of  $p$  into  $q$ .

The term co-product may be applied to such expressions, regarded as an abbreviation merely, after the analogy of cosine for complement of the sine.

Consider next a unit normal vector system. By the definition we have

$$|z_1 = z_2, \quad |z_2 = |(|z_1) = -z_1,$$

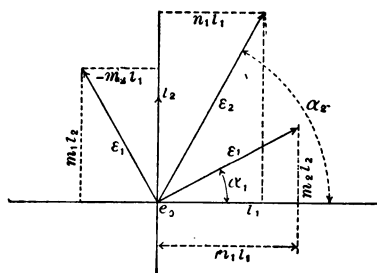
$$\text{because } z_1 | z_1 = z_1 z_2 = I,$$

$$z_2 | z_2 = z_2 (-z_1) = -z_2 z_1 = z_1 z_2 = I.$$

$$\text{Also, } z_1 | z_2 = -z_1 z_1 = 0 = z_2 | z_1.$$

Next let

$$\epsilon_1 = m_1 z_1 + m_2 z_2 \quad \text{and} \quad \epsilon_2 = n_1 z_1 + n_2 z_2;$$



\* Grassmann (1862), Chapter 4.

then, by (b) and (c),

$$|\epsilon_1 = m_1|z_1 + m_2|z_2 = m_1z_2 - m_2z_1. \quad (53)$$

By the figure it is evident that  $|\epsilon_1$  is a vector of the same length as  $\epsilon_1$  and perpendicular to it, or, in other words, taking the complement of a vector in plane space rotates it positively through  $90^\circ$ .

The co-product  $\epsilon_1|\epsilon_2$  is the area of the parallelogram, two of whose sides are  $\epsilon_1$  and  $|\epsilon_2$ , drawn outwards from a point; if  $\epsilon_1$  is parallel to  $|\epsilon_2$ , this area vanishes, or  $\epsilon_1|\epsilon_2 = 0$ ; but, since  $|\epsilon_2$  is perpendicular to  $\epsilon_2$ ,  $\epsilon_1$  must in this case be perpendicular to  $\epsilon_2$ ; hence the equation

$$\epsilon_1|\epsilon_2 = 0 \quad (54)$$

is the condition that two vectors  $\epsilon_1$  and  $\epsilon_2$  shall be perpendicular to each other.

The co-product  $\epsilon_1|\epsilon_1$ , which will usually be written  $\epsilon_1^2$ , and called the co-square of  $\epsilon_1$ , is the area of a square each of whose sides has the length  $T\epsilon_1$ ; hence

$$T\epsilon_1 = \sqrt{\epsilon_1|\epsilon_1} = \sqrt{\epsilon_1^2}. \quad (55)$$

Let  $\alpha_1$  and  $\alpha_2$  be the angles between  $z_1$  and  $\epsilon_1$  and between  $z_1$  and  $\epsilon_2$  respectively, as in the figure. Then

$$\epsilon_1\epsilon_2 = m_1n_2 - m_2n_1 = T\epsilon_1T\epsilon_2 \sin(\alpha_2 - \alpha_1), \quad (56)$$

the third member being the ordinary expression for the area of the parallelogram  $\epsilon_1\epsilon_2$ . Also

$$\begin{aligned} \epsilon_1|\epsilon_2 &= (m_1z_1 + m_2z_2)(n_1z_2 - n_2z_1) \\ &= m_1n_1 + m_2n_2 = T\epsilon_1T\epsilon_2 \cos(\alpha_2 - \alpha_1), \end{aligned} \quad (57)$$

the last member being found as before, remembering that  $\sin(90^\circ + \alpha_2 - \alpha_1) = \cos(\alpha_2 - \alpha_1)$ .

If in (57) we let  $\epsilon_2 = \epsilon_1$ , whence  $n_1 = m_1$  and  $n_2 = m_2$ , we have

$$T\epsilon_1 = \sqrt{\epsilon_1^2} = \sqrt{m_1^2 + m_2^2}. \quad (58)$$

If  $T\epsilon_1 = T\epsilon_2 = 1$ , then  $m_1 = \cos \alpha_1$ ,  $m_2 = \sin \alpha_1$ ,  $n_1 = \cos \alpha_2$ ,  $n_2 = \sin \alpha_2$ , and equations (56) and (57) give the ordinary trigonometrical formulas  $\sin(\alpha_2 - \alpha_1) = \sin \alpha_2 \cos \alpha_1 - \cos \alpha_2 \sin \alpha_1$ ,

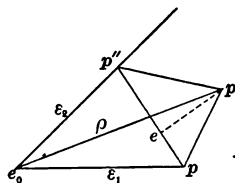
and  $\cos(\alpha_2 - \alpha_1) = \cos \alpha_1 \cos \alpha_2 + \sin \alpha_1 \sin \alpha_2$ . Squaring and adding (56) and (57), there results

$$T^2 \epsilon_1 \cdot T^2 \epsilon_2 = \epsilon_1^2 \epsilon_2^2 = (\epsilon_1 \epsilon_2)^2 + (\epsilon_1 | \epsilon_2)^2. \quad (59)$$

Attention is called to the fact, which the student may have already noticed, that such an equation as  $AB = AC$ , in which  $AB$  and  $AC$  are combinatory products, does not, in general, imply that  $B = C$ , for the reason that the equation  $A(B - C) = 0$  can usually be satisfied without either factor being itself zero. Thus  $pL_1 = pL_2$  means simply that the two quantities which are equated have the same magnitude and sign, which permits  $L_2$  to have an infinity of lengths and positions, when  $p$  and  $L_1$  are given. The equation  $p_1 p_2 = p_1 p_3$ , or  $p_1(p_2 - p_3) = 0$ ,  $p_2$  and  $p_3$  being unit points, implies, however, that  $p_2 = p_3$ , unless  $p_1$  is at  $\infty$ , that is, a vector.

Exercise 9.—A triangle whose sides are of constant length moves so that two of its vertices remain on two fixed lines: find the locus of the other vertex.

Let  $e_0 \epsilon_1$  and  $e_0 \epsilon_2$  be the two fixed lines, and  $pp'p''$  the triangle. Let  $pe$  be perpendicular to  $p'p''$ ,  $p' - e_0 = x\epsilon_1$  and  $p'' - e_0 = y\epsilon_2$ ; then  $p'' - p' = y\epsilon_2 - x\epsilon_1$ ,  $T(y\epsilon_2 - x\epsilon_1) = c = \text{constant}$ , by the conditions. Also,  $Tp'e = \text{constant} = mc$ , say, and  $Te p = \text{constant} = nc$ , say. Hence



$$e - p' = Tp'e. U(e - p') = mc \cdot \frac{y\epsilon_2 - x\epsilon_1}{T(y\epsilon_2 - x\epsilon_1)} = m(y\epsilon_2 - x\epsilon_1),$$

and similarly  $p - e = n(y\epsilon_2 - x\epsilon_1)$ . Therefore

$$p - e_0 = \rho = x\epsilon_1 + m(y\epsilon_2 - x\epsilon_1) + n(y\epsilon_2 - x\epsilon_1),$$

an equation which, with the condition  $T(y\epsilon_2 - x\epsilon_1) = c$ , or

$$y^2 \epsilon_2^2 - 2xy\epsilon_1 | \epsilon_2 + x^2 \epsilon_1^2 = c^2,$$

determines the locus to be a second-degree curve, which must in fact be an ellipse, since it can have no points at infinity. Let us rearrange the equation in  $\rho$  thus:

$$\rho = x[(1 - m)\epsilon_1 - n | \epsilon_1] + y[m\epsilon_2 + n | \epsilon_2] = x\epsilon + y\epsilon', \text{ say,}$$



so that  $\epsilon = (1 - m)\epsilon_1 - n|\epsilon_1$  and  $\epsilon' = m\epsilon_2 + n|\epsilon_2$ ; then multiply successively into  $\epsilon$  and  $\epsilon'$ ; therefore  $\rho\epsilon = y\epsilon'\epsilon$  and  $\rho\epsilon' = x\epsilon\epsilon'$ . Substituting these values of  $x$  and  $y$  in the equation of condition, we have

$$\epsilon_1^2 \cdot (\rho\epsilon)^2 + 2\epsilon_1|\epsilon_2 \cdot \rho\epsilon \cdot \rho\epsilon' + \epsilon_1^2(\rho\epsilon')^2 = c^2(\epsilon\epsilon')^2,$$

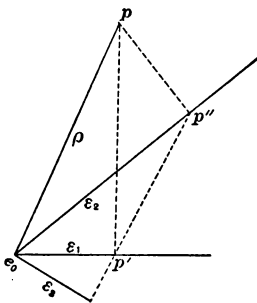
a scalar equation of the second degree in  $\rho$ .

**Exercise 10.**—There is given an irregular polygon of  $n$  sides: show that if forces act at the middle points of these sides, proportional to them in magnitude, and directed all outward or else all inward, these forces will be in equilibrium.

Let  $e_0$  be a vertex of the polygon, and let  $2\epsilon_1, 2\epsilon_2, \dots, 2\epsilon_n$  represent its sides in magnitude and direction. Then the middle points will be  $e_0 + \epsilon_1, e_0 + 2\epsilon_1 + \epsilon_2$ , etc., and, using the complement in a vector system, we have

$$\begin{aligned} \sum \rho\epsilon &= (e_0 + \epsilon_1)|\epsilon_1 + (e_0 + 2\epsilon_1 + \epsilon_2)|\epsilon_2 + (e_0 + 2\epsilon_1 + 2\epsilon_2 + \epsilon_3)|\epsilon_3 + \dots \\ &\quad + (e_0 + 2\epsilon_1 + \dots + 2\epsilon_{n-1} + \epsilon_n)|\epsilon_n \\ &= e_0 \left| \sum_1^n \epsilon + \sum_1^n \epsilon^2 + 2\epsilon_1 \right| \sum_2^n \epsilon + 2\epsilon_2 \left| \sum_3^n \epsilon + \dots + 2\epsilon_{n-1} \right| \epsilon_n \\ &= e_0 \left| \sum_1^n \epsilon + \left( \sum_1^n \epsilon \right)^2 \right| = 0, \text{ which was to be proved.} \end{aligned}$$

**Exercise 11.**—A line passes through a fixed point and cuts two fixed lines; at the points of intersection perpendiculars to the fixed lines are erected; find the locus of the intersection of these perpendiculars.



Let the fixed lines be  $e_0\epsilon_1$  and  $e_0\epsilon_2$ , and the fixed point  $e_0 + \epsilon_3$ ; the moving line cuts the fixed lines in  $p'$  and  $p''$ , at which points perpendiculars are erected meeting in  $p$ .

Let  $p - e_0 = \rho$ ,  $p' - e_0 = x\epsilon_1$ ,  $p'' - e_0 = y\epsilon_2$ ,  $T\epsilon_1 = T\epsilon_2 = 1$ ; then  $\rho = x\epsilon_1 + x'|\epsilon_1 = y\epsilon_2 + y'|\epsilon_2$ , whence  $\rho|\epsilon_1 = x$  and  $\rho|\epsilon_2 = y$ .

Also, since  $e_0 + \epsilon_1, p', p''$  are collinear points,

$$(x\epsilon_1 - \epsilon_1)(y\epsilon_2 - \epsilon_2) = 0 = xy\epsilon_1\epsilon_2 + y\epsilon_2\epsilon_1 + x\epsilon_1\epsilon_1;$$

or, substituting values of  $x$  and  $y$ ,

$$\rho|\epsilon_1 \cdot \rho|\epsilon_2 \cdot \epsilon_1\epsilon_2 + \rho|\epsilon_2 \cdot \epsilon_2\epsilon_1 + \rho|c_1 \cdot \epsilon_2\epsilon_1 = 0,$$

an equation of the second degree in  $\rho$ , and hence representing a conic.

Prob. 15. If  $a, b, c$  are the lengths of the sides of a triangle, prove the formula  $a^2 = b^2 + c^2 - 2bc \cos A$ , by taking vectors  $\epsilon_1, \epsilon_2$ , and  $\epsilon_2 - \epsilon_1$  equal to the respective sides.

Prob. 16. If  $e_0\epsilon_1$  and  $e_0\epsilon_2$  are two unit lines, show that the vector perpendicular from  $e_0$  on the line  $(e_0 + a\epsilon_1)(e_0 + b\epsilon_2)$  is

$\frac{ab\epsilon_1\epsilon_2}{(b\epsilon_2 - a\epsilon_1)^2} \cdot (b\epsilon_2 - a\epsilon_1)$ , of which the length is  $\frac{ab\epsilon_1\epsilon_2}{T(b\epsilon_2 - a\epsilon_1)}$ . From this derive the Cartesian expression for the perpendicular from the origin upon a straight line in oblique coordinates,

$ab \sin \omega \div (a^2 + b^2 - 2ab \cos \omega)^{1/2}$ ,  $\omega$  being angle between the axes.

Prob. 17. If three points,  $me_0 + ne_1, me_1 + ne_2, me_2 + ne_0$ , be taken on the sides of the reference triangle, then the sides of the complementary triangle,  $(me_0 + ne_1)$ , etc., will be respectively parallel to the corresponding sides of the triangle formed by the assumed points  $(me_1 + ne_2), (me_2 + ne_0)$ , etc.

## ART. 9. EQUATIONS OF CONDITION, AND FORMULAS.

Several equations of condition are placed here together for convenient reference: some have been already given; others follow from the results of Arts. 7 and 8. When we have

$$\begin{array}{l|l} p_1 p_2 = 0, \} & L_1 L_2 = 0, \} \\ \text{or } n_1 p_1 + n_2 p_2 = 0, \} & \text{or } n_1 L_1 + n_2 L_2 = 0, \} \end{array} \quad (60)$$

the two points coincide;

the two lines coincide;

$$\begin{array}{l|l} p_1 p_2 p_3 = 0, \} & L_1 L_2 L_3 = 0, \} \\ \text{or } \sum_1 n p = 0, \} & \text{or } \sum_1 n L = 0, \} \end{array} \quad (61)$$

the three points are collinear;

the three lines are confluent.

$$\epsilon_1 \epsilon_2 = 0, \quad \text{or} \quad n_1 \epsilon_1 + n_2 \epsilon_2 = 0, \quad (62)$$

the two vectors are parallel (points at infinity coincide);

$$\epsilon_1 | \epsilon_2 = 0, \quad (63)$$

the two vectors are perpendicular ;

$$\begin{array}{l|l} \rho_1 | \rho_2 = 0, & L_1 | L_2 = 0, \quad (64) \\ \text{either point lies on the com-} & \text{either line passes through the} \\ \text{plementary line of the other.} & \text{complementary point of the} \\ & \text{other.} \end{array}$$

If we write the equation

$$\rho = x_1 \epsilon_1 + x_2 \epsilon_2,$$

$x_1 \epsilon_1$  is the projection of  $\rho$  on  $\epsilon_1$  parallel to  $\epsilon_2$ , and  $x_2 \epsilon_2$  is the projection of  $\rho$  on  $\epsilon_2$  parallel to  $\epsilon_1$ . Multiply both sides of the equation into  $\epsilon_2$ ; therefore  $\rho \epsilon_2 = x_1 \epsilon_1 \epsilon_2$ , or  $x_1 = \rho \epsilon_2 \div \epsilon_1 \epsilon_2$ . Similarly, multiplying into  $\epsilon_1$ , we have  $\rho \epsilon_1 = x_2 \epsilon_2 \epsilon_1$ , or  $x_2 = \rho \epsilon_1 \div \epsilon_2 \epsilon_1$ , whence

$$\rho = \frac{\epsilon_1 \cdot \rho \epsilon_2}{\epsilon_1 \epsilon_2} + \frac{\epsilon_2 \cdot \rho \epsilon_1}{\epsilon_2 \epsilon_1}. \quad (65)$$

The two terms of the second member of (65) are therefore the projections of  $\rho$  on  $\epsilon_1$  parallel to  $\epsilon_2$ , and on  $\epsilon_2$  parallel to  $\epsilon_1$ , respectively.\*

Let  $\epsilon_1$  and  $\epsilon_2$  be unit normal vectors, say,  $z$  and  $|z$ ; then (65) becomes

$$\rho = z \cdot \rho |z - |z \cdot \rho z = z \cdot \rho |z + |z \cdot \rho z; \quad (66)$$

or, if  $z_1$  and  $z_2$  be used instead of  $z$  and  $|z$ ,

$$\rho = z_1 \cdot \rho |z_1 + z_2 \cdot \rho |z_2. \quad (67)$$

Again, in (65) let  $\rho = \epsilon_3$ , clear of fractions, and transpose ; therefore

$$\epsilon_1 \epsilon_2 \cdot \epsilon_3 + \epsilon_2 \epsilon_3 \cdot \epsilon_1 + \epsilon_3 \epsilon_1 \cdot \epsilon_2 = 0, \quad (68)$$

a symmetrical relation between any three directions in plane space. Let  $T\epsilon_1 = T\epsilon_2 = T\epsilon_3 = 1$ , and multiply (68) into  $|\epsilon_3$ , thus

$$\epsilon_1 \epsilon_2 + \epsilon_2 \epsilon_3 \cdot \epsilon_1 |\epsilon_3 + \epsilon_3 \epsilon_1 \cdot \epsilon_2 |\epsilon_3 = 0, \quad (69)$$

which is equivalent to

$$\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta,$$

the upper or lower sign corresponding to the case when  $\epsilon_3$  is

\* Grassmann (1844), Chapter 5 (1862), Art. 129. Hyde's Directional Calculus, Arts. 46 and 47.

between  $\epsilon_1$  and  $\epsilon_2$ , or outside, respectively. Writing in (69)  $|\epsilon_2$  instead of  $\epsilon_2$ , we have

$$\epsilon_1|\epsilon_2 - \epsilon_2|\epsilon_1 \cdot \epsilon_1|\epsilon_1 + \epsilon_2\epsilon_1 \cdot \epsilon_2\epsilon_1 = 0, \quad (70)$$

which gives the  $\cos(\alpha \pm \beta)$ . These formulas being for any three directions in plane space, are independent of the magnitude of the angles involved.

There is given below a set of formulas for points and lines, arranged in complementary pairs, and all placed together for convenient reference, the derivation of them following after.

$$\begin{aligned} p &= (p_0 p_1 p_2)^{-1} [p_0 \cdot p p_1 p_2 + p_1 \cdot p p_2 p_0 + p_2 \cdot p p_0 p_1], \\ L &= (L_0 L_1 L_2)^{-1} [L_0 \cdot L L_1 L_2 + L_1 \cdot L L_2 L_0 + L_2 \cdot L L_0 L_1] \end{aligned} \quad (71)$$

$$\begin{aligned} p &= (p_0 p_1 p_2)^{-1} [p_1 p_2 \cdot p | p_0 + p_2 p_0 \cdot p | p_1 + p_0 p_1 \cdot p | p_2], \\ L &= (L_0 L_1 L_2)^{-1} [L_1 L_2 \cdot L | L_0 + L_2 L_0 \cdot L | L_1 + L_0 L_1 \cdot L | L_2] \end{aligned} \quad (72)$$

$$\left. \begin{aligned} p_1 p_2 \cdot p_3 p_4 &= -p_1 \cdot p_2 p_3 p_4 + p_2 \cdot p_3 p_4 p_1 \\ &= p_3 \cdot p_4 p_1 p_2 - p_4 \cdot p_1 p_2 p_3, \\ L_1 L_2 \cdot L_3 L_4 &= -L_1 \cdot L_2 L_3 L_4 + L_2 \cdot L_3 L_4 L_1 \\ &= L_3 \cdot L_4 L_1 L_2 - L_4 \cdot L_1 L_2 L_3 \end{aligned} \right\}, \quad (73)$$

$$p_1 p_2 \cdot |q_1 = - \left| \begin{matrix} p_1 & p_1 | q_1 \\ p_2 & p_2 | q_1 \end{matrix} \right|, \quad L_1 L_2 | M_1 = - \left| \begin{matrix} L_1 & L_1 | M_1 \\ L_2 & L_2 | M_1 \end{matrix} \right|, \quad (74)$$

$$p_2 | q_1 q_2 = \left| \begin{matrix} |q_1 & p_2 | q_1 \\ |q_2 & p_2 | q_2 \end{matrix} \right|, \quad L_2 | M_1 M_2 = \left| \begin{matrix} |M_1 & L_2 | M_1 \\ |M_2 & L_2 | M_2 \end{matrix} \right|, \quad (75)$$

$$p_1 p_2 | q_1 q_2 = \left| \begin{matrix} p_1 | q_1 & p_1 | q_2 \\ p_2 | q_1 & p_2 | q_2 \end{matrix} \right|, \quad L_1 L_2 | M_1 M_2 = \left| \begin{matrix} L_1 | M_1 & L_1 | M_2 \\ L_2 | M_1 & L_2 | M_2 \end{matrix} \right|, \quad (76)$$

$$p_0 p_1 p_2 \cdot q_0 q_1 q_2 = \left| \begin{matrix} p_0 | q_0 & p_0 | q_1 & p_0 | q_2 \\ p_1 | q_0 & p_1 | q_1 & p_1 | q_2 \\ p_2 | q_0 & p_2 | q_1 & p_2 | q_2 \end{matrix} \right| \quad (77)$$

The complementary formula to (77) is not given, but may be obtained by putting  $L$ 's and  $M$ 's for  $p$ 's and  $q$ 's.

Derivation of Equations (71)-(77).—Equation (71). Write  $p = x_0 p_0 + x_1 p_1 + x_2 p_2$ , and multiply this equation by  $p_1 p_2$ ; then  $p_1 p_2 p = x_0 p_1 p_2 p_0$ , or  $x_0 = p p_1 p_2 \div p_0 p_1 p_2$ .

Multiplying similarly by  $p_0 p_2$  and by  $p_0 p_1$ , we find

$x_1 = p p_2 p_0 \div p_0 p_1 p_2$  and  $x_2 = p p_0 p_1 \div p_0 p_1 p_2$ . The substitu-

tion of these values gives the first of (71), and the second is similarly obtained or may be found by simply putting  $L$ 's for  $p$ 's in the first.

Equation (72). Write  $p = x_0 |p_1 p_2 + x_1 |p_2 p_0 + x_2 |p_0 p_1$ , and multiply into  $|p_0$ ; thus  $p |p_0 = x_0 p_0 p_1 p_2$ . Find in the same way values of  $x_1$  and  $x_2$ , and substitute.

Equation (73). Write  $p_1 p_2 \cdot p_3 p_4 = xp_1 + yp_2$ , and multiply by  $pp_2$ ; therefore  $pp_2 \cdot p_1 p_2 \cdot p_3 p_4 = xpp_2 p_1$ , or, by Eq. (38),  $p_2 pp_1 \cdot p_2 p_3 p_4 = xpp_2 p_1 = -xp_2 pp_1$ ; or,  $x = -p_2 p_3 p_4$ . Multiplying by  $pp_1$  we find  $y = p_2 p_3 p_1$ , and on substituting obtain the first of (73). For the second put  $p_1 p_2 \cdot p_3 p_4 = xp_2 + yp_1$ , and proceed in a similar way.

Equation (74). In the first of (73) put  $p_2 p_4 = |q_1$ .

Equation (75). In the fourth of (73) put

$$L_1 L_2 = p_2, L_3 = |q_1, L_4 = |q_2.$$

Equation (76). Multiply (75) by  $p_1$ .

Equation (77). In the first of (72) put  $q_2$  for  $p$ , and multiply by  $p_0 p_1 p_2 \cdot q_0 q_1$ ; then

$$\begin{aligned} p_0 p_1 p_2 \cdot q_0 q_1 q_2 &= q_0 q_1 |p_1 p_2 \cdot q_2 |p_0 + q_0 q_1 |p_2 p_0 \cdot q_2 |p_1 + q_0 q_1 |p_0 p_1 \cdot q_2 |p_2 \\ &= p_0 |q_2 \cdot \begin{vmatrix} p_1 |q_0 & p_1 |q_1 \\ p_2 |q_0 & p_2 |q_1 \end{vmatrix} + p_1 |q_2 \cdot \begin{vmatrix} p_2 |q_0 & p_2 |q_1 \\ p_0 |q_0 & p_0 |q_1 \end{vmatrix} + p_2 |q_2 \cdot \begin{vmatrix} p_0 |q_0 & p_0 |q_1 \\ p_1 |q_0 & p_1 |q_1 \end{vmatrix}, \end{aligned}$$

by (76), which is equivalent to the third order determinant of equation (77).\*

Exercise 12.—To show the product of two determinants as a determinant of the same order.

Let  $p_0 = \sum_0^2 l e$ ,  $p_1 = \sum m e$ ,  $p_2 = \sum n e$ ,  $q_0 = \sum \lambda e$ ,  $q_1 = \sum \mu e$ ,  $q_2 = \sum v e$ ;

then  $p_0 p_1 p_2 = [l_0, m_1, n_2]$ ,  $q_0 q_1 q_2 = [\lambda_0, \mu_1, v_2]$ ; also  $p_0 |q_0 = l_0 \lambda_0 + l_1 \lambda_1 + l_2 \lambda_2$ ,  $p_1 |q_0 = m_0 \lambda_0 + m_1 \lambda_1 + m_2 \lambda_2$ , etc. Substituting these values in (77), we have the required result. A solution may also be obtained directly without the use of (77).

Let the  $q$ 's be as above, but write  $p_0 = \sum_0^2 l q$ ,  $p_1 = \sum m q$ ,  $p_2 = \sum n q$ .

Then

$$p_0 p_1 p_2 = \sum l q \cdot \sum m q \cdot \sum n q = [l_0, m_1, n_2] q_0 q_1 q_2 = [l_0, m_1, n_2] [\lambda_0, \mu_1, v_2].$$

\* Grassmann (1862), Art. 173.

Also  $p_0 = l_0 \Sigma \lambda e + l_1 \Sigma \mu e + l_2 \Sigma \nu e$

$$= (l_0 \lambda_0 + l_1 \mu_0 + l_2 \nu_0) e_0 + (l_0 \lambda_1 + l_1 \mu_1 + l_2 \nu_1) e_1 + (l_0 \lambda_2 + l_1 \mu_2 + l_2 \nu_2) e_2,$$

with similar values for  $p_1$  and  $p_2$ , which on being substituted in  $p_0 p_1 p_2$  give the result. Equation (77), however, exhibits the product in a very compact, symmetrical, and easily remembered form.\*

Exercise 13.—Show that the sides  $p_1 p_2$ ,  $p_2 p_3$ ,  $p_3 p_1$  of the triangle  $p_1 p_2 p_3$  cut the corresponding sides  $|p_3|$ ,  $|p_1|$ ,  $|p_2|$  of the complementary triangle in three collinear points.

The three points of intersection are, using (74),

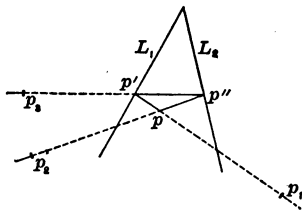
$$p_1 p_2 \cdot |p_3| = -p_1 \cdot p_2 |p_3 + p_1 \cdot p_1 |p_3|, \quad p_2 p_3 \cdot |p_1| = -p_2 \cdot p_3 |p_1 + p_2 \cdot p_2 |p_1|,$$

$$p_3 p_1 \cdot |p_2| = -p_3 \cdot p_1 |p_2 + p_1 \cdot p_3 |p_2|, \text{ of which the sum is zero,}$$

showing that the points are collinear. It may be shown in the same way that the lines joining corresponding vertices are confluent.

Exercise 14.—If the sides of a triangle pass through three fixed points, and two of the vertices slide on fixed lines, find the locus of the other vertex.

Let the fixed points and lines be  $p_1, p_2, p_3, L_1, L_2$ , and  $p, p', p''$  the vertices of the triangle, as in the figure. Then  $p' p_3 p'' = 0$ ;  $p'$  coincides with  $p p_1 \cdot L_1$  and  $p''$  with  $p p_2 \cdot L_2$ ; hence substituting  $(p p_1 \cdot L_1) p_3 (L_2 \cdot p_2 p) = 0$ , the equation of the locus, which, being of the second degree in  $p$ , is that of a conic.



Prob. 18. Show that if the three fixed points of the last exercise are collinear, then the locus of  $p$  breaks up into two straight lines. Use equation (73).

Prob. 19. If the vertices of a triangle slide on three fixed lines, and two of the sides pass through fixed points, find the envelope of the other side. (This statement is reciprocally related to that of Exercise 14, that is, lines and points are replaced by points and

\* These methods may be applied to determinants of any order by using a space of corresponding order.

lines respectively, and the resulting equation will be an equation of the second order in  $L$ , a variable line.)

Prob. 20. Show that if the three fixed lines of Problem 19 are confluent, then the envelope of  $L$  reduces to two points and the line joining them.

#### ART. 10. STEREOMETRIC PRODUCTS.

The product of two points in solid space is the same as in plane space. See Art. 7.

Product of Three Points.—Any three points determine a plane, and also, as in Art. 7, an area; hence  $p_1 p_2 p_3$  is a plane-sect or a portion of the plane fixed by the three points whose area is double that of the triangle  $p_1 p_2 p_3$ . It may be shown, in the manner used in Art. 7 for the sect, that no plane-sect, not in this plane, can be equal to  $p_1 p_2 p_3$ , and that any plane-sect in this plane having the same area and sign will be equal to  $p_1 p_2 p_3$ .<sup>\*</sup> Of course  $p_1 p_2 p_3$  is not now scalar.

Product of Four Points.—Any four non-coplanar points determine a tetrahedron, say  $p_1 p_2 p_3 p_4$ , and six times the volume of this tetrahedron is taken for the value of the product, because this is the volume of the parallelepiped generated by the product  $p_1 p_2 p_3$ ,—i.e. the parallelogram  $p_1, p_2$ ,—when it moves parallel to its initial position from  $p_1$  to  $p_4$ . Let  $p_2 - p_1 = \epsilon$ ,  $p_3 - p_1 = \epsilon'$ ,  $p_4 - p_1 = \epsilon''$ , then

$$p_1 p_2 p_3 p_4 = p_1 p_2 p_3 \epsilon'' = p_1 p_3 \epsilon' \epsilon'' = p_1 \epsilon \epsilon' \epsilon''. \quad (78)$$

If  $p_1 = \sum_0^3 k e$ ,  $p_2 = \sum_0^3 l e$ ,  $p_3 = \sum_0^3 m e$ ,  $p_4 = \sum_0^3 n e$ , then

$$p_1 p_2 p_3 p_4 = \sum k e \sum l e \sum m e \sum n e = [k, l, m, n] \cdot e_0 e_1 e_2 e_3; \quad (79)$$

from which it appears that any two quadruple products of points differ from each other only by a scalar factor, that is, they differ only in magnitude, or sign, or both; hence such products are themselves scalar.† If  $p_1 p_2 p_3 p_4 = 0$ , the volume of the tetrahedron vanishes, so that the four points are coplanar.

<sup>\*</sup> Grassmann (1862), Art. 255.

† Grassmann (1862), Art. 263.

**Product of Two Vectors.**—The two vectors determine an area as in Art. 7, but they also determine now a plane direction, so that the product  $\epsilon_1\epsilon_2$  is a plane-vector, and is not scalar as in plane space. Also,  $\epsilon_1\epsilon_2$  differs from  $p_1\epsilon_1\epsilon_2$  now just as  $\epsilon$  differs from  $p\epsilon$ ; namely,  $\epsilon_1\epsilon_2$  has a definite area and plane direction, that is, toward a certain line at infinity, while  $p_1\epsilon_1\epsilon_2$  is fixed in position by passing through  $p_1$ . Equation (37) therefore does not hold in solid space.

**Product of Three Vectors.**—Three vectors determine a parallelepiped as in the figure above, and  $\epsilon\epsilon'\epsilon''$  is therefore the volume of this parallelepiped. Any other triple vector product can differ from this only in magnitude and sign. For let  $\epsilon_1\epsilon_2\epsilon_3$  be such a product, and write

$$\epsilon = x_1\epsilon_1 + x_2\epsilon_2 + x_3\epsilon_3, \epsilon' = \sum_1^3 y_i\epsilon_i, \epsilon'' = \sum_1^3 z_i\epsilon_i; \text{ then}$$

$$\epsilon\epsilon'\epsilon'' = \sum x_i \sum y_j \sum z_k \epsilon_i \epsilon_j \epsilon_k = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} \epsilon_1\epsilon_2\epsilon_3, \quad (80)$$

so that the two products only differ by the scalar determinant factor. Hence the product of three vectors must be itself a scalar, by Art. 1. Since, then, the product of four points has precisely the same signification as that of three vectors, we may write

$$\begin{aligned} p_1p_2p_3p_4 &= p_1\epsilon\epsilon'\epsilon'' = \epsilon\epsilon'\epsilon'' = (p_2 - p_1)(p_3 - p_1)(p_4 - p_1) \\ &= p_2p_3p_4 - p_3p_4p_1 + p_4p_1p_2 - p_1p_2p_3. \end{aligned} \quad (81)$$

Thus the sum of the plane-sects forming the doubles of the faces of a tetrahedron, all taken positively in the same sense as looked at from outside the tetrahedron, is equal to the volume of the tetrahedron. Compare equation (37).

If  $\epsilon\epsilon'\epsilon'' = 0$ , the volume of the parallelepiped vanishes, and the three vectors must be parallel to one plane.

**Product of Two Sects.**—In solid space two sects determine a tetrahedron of which they are opposite edges. Thus

$$p_1p_2p_3p_4 = p_1p_2 \cdot p_3p_4 = L_1L_2 = p_3p_4 \cdot p_1p_2 = L_2L_1, \quad (82)$$

so that the stereometric product of two sects is commutative, and has the same meaning as that of four points.





Product of Two Plane-Vectors.—Let  $\eta_1$  and  $\eta_2$  be two plane-vectors or lines at infinity; let  $\epsilon$  be parallel to each of them, and  $\epsilon_1$  and  $\epsilon_2$  so taken that  $\eta_1 = \epsilon\epsilon_1$ ,  $\eta_2 = \epsilon\epsilon_2$ , then

$$\eta_1\eta_2 = \epsilon\epsilon_1 \cdot \epsilon\epsilon_2 = \epsilon\epsilon_1\epsilon_2 \cdot \epsilon = -\eta_2\eta_1, \quad (89)$$

because  $\eta_1$  and  $\eta_2$  determine a common direction  $\epsilon$ , and a parallelepiped of which three conterminous edges are equal to  $\epsilon$ ,  $\epsilon_1$ ,  $\epsilon_2$ , respectively.

Product of Three Plane-Vectors.—Take  $\epsilon_1$ ,  $\epsilon_2$ ,  $\epsilon_3$  so that

$$\eta_1\eta_2\eta_3 = n \cdot \epsilon_3\epsilon_2 \cdot \epsilon_2\epsilon_1 \cdot \epsilon_1\epsilon_3 = n(\epsilon_1\epsilon_2\epsilon_3)^2. \quad (90)$$

The directions  $\epsilon_1 \dots \epsilon_3$  are common to the plane-vectors  $\eta_1 \dots \eta_3$  taken two by two.

Several conditions are given here together which follow from the results of this article.

$p_1p_2 = 0,$	$P_1P_2 = 0,$	(91)
Two points coincide.	Two planes coincide.	

$p_1p_2p_3 = 0,$	$P_1P_2P_3 = 0,$	(92)
Three points collinear.	Three planes collinear.	

$p_1p_2p_3p_4 = p_1p_2 \cdot p_3p_4$	$P_1P_2P_3P_4 = P_1P_2 \cdot P_3P_4$	
$= L_1L_2 = 0,$	$= L_1L_2 = 0,$	(93)
Four points coplanar; two lines intersect.	Four planes confluent; two lines intersect.	

$\epsilon_1\epsilon_2 = 0,$	$\eta_1\eta_2 = 0,$	(94)
Vectors parallel.	Plane-vectors parallel.	

$\epsilon_1\epsilon_2\epsilon_3 = 0,$	$\eta_1\eta_2\eta_3 = 0,$	(95)
Three vectors parallel to one plane.	Three plane-vectors parallel to one line.	

Sum of Two Planes.—Let them be  $P_1$  and  $P_2$ , let  $L$  be a sect in their common line, and take  $p_1$  and  $p_2$  so that  $P_1 = Lp_1$ ,  $P_2 = Lp_2$ ; then

$$P_1 + P_2 = L(p_1 + p_2) = 2L\bar{p}, \quad (96)$$

$\bar{p}$  being the mean of  $p_1$  and  $p_2$ . Also

$$P_1 - P_2 = L(p_1 - p_2); \quad (97)$$

whence the sum and difference are the diagonal plane through  $L$ , and a plane through  $L$  parallel to the diagonal plane which is itself parallel to  $L$ , of the parallelepiped determined by  $P_1$

and  $P_2$ . If  $TP_1 = TP_2$ ,  $P_1 \pm P_2$  will evidently be the two bisecting planes of the angle between them. The bisecting planes may also be written

$$\frac{P_1}{TP_1} \pm \frac{P_2}{TP_2} \quad \text{or} \quad P_1 TP_2 \pm P_2 TP_1. \quad (98)$$

If the two planes are parallel, let  $\eta$  be a plane-vector parallel to each of them, that is, their common line at infinity, and let  $p_1$  and  $p_2$  be points in the respective planes; then we may write  $P_1 = n_1 p_1 \eta$ ,  $P_2 = n_2 p_2 \eta$ , whence

$$P_1 + P_2 = (n_1 p_1 + n_2 p_2) \eta = (n_1 + n_2) \bar{p} \eta. \quad (99)$$

If  $n_1 + n_2 = 0$ , this becomes

$$P_1 + P_2 = n_2 (p_2 - p_1) \eta, \quad (100)$$

the product of a vector into a plane-vector and therefore a scalar, by (80).

Two plane-vectors may be added similarly, since they will have a common direction, namely, that of the vector parallel to both of them.

Exercise 15.—If two tetrahedra  $e_0 e_1 e_2 e_3$  and  $e'_0 e'_1 e'_2 e'_3$  are so situated that the right lines through the pairs of corresponding vertices all meet in one point, then will the corresponding faces cut each other in four coplanar lines.

The given conditions are equivalent to  $e_0 e'_0 \cdot e_1 e'_1 = 0$   
 $= e_0 e'_0 \cdot e_2 e'_2 = e_0 e'_0 \cdot e_3 e'_3 = e_1 e'_1 \cdot e_2 e'_2 = e_1 e'_1 \cdot e_3 e'_3 = e_2 e'_2 \cdot e_3 e'_3 = e_2 e'_2 \cdot e_1 e'_1$ .  
 Two of the intersecting lines of faces are  $e_0 e_1 e_2 \cdot e'_0 e'_1 e'_2$  and  $e_1 e_2 e_3 \cdot e'_1 e'_2 e'_3$ , and, if these intersect, we must accordingly have, by (93),  $012 \cdot 0'1'2' \cdot 123 \cdot 1'2'3' = 0 = 012 \cdot 123 \cdot 0'1'2' \cdot 1'2'3' = 0123 \cdot 0'1'2'3' \cdot 121'2'$ , the last factor of which is equivalent to the fourth condition above, since quadruple-point products in solid space are associative. Similarly all the other pairs of intersections may be treated.

Exercise 16.—The twelve bisecting planes of the dihedral angles of a tetrahedron fix eight points, the centers of the inscribed and escribed spheres, through which they pass six by six.

The sum and difference of two unit planes are their two

bisecting planes, by (97). Let the tetrahedron be  $e_0e_1e_2e_3$ , and let the double areas of its faces be  $A_0 = Te_1e_2e_3$ , etc.; then a pair of bisecting planes will be  $\frac{e_0e_1e_2}{A_0} \pm \frac{e_0e_1e_3}{A_0}$  or  $e_0e_1(A_2e_2 \pm A_3e_3)$ . The pair through the opposite edge will be  $e_2e_3(A_0e_0 \pm A_1e_1)$ . If there be a point through which the six internal bisecting planes pass, it must be on the intersection of these two planes taken with the upper signs, and we infer by symmetry that it must be the point  $\sum_0^3 Ae$ . Another internal bisecting plane is  $e_1e_0(A_2e_2 + A_3e_3)$ , which gives zero when multiplied into  $\sum Ae$ , as do also the other three.

To obtain all the points we have only to use the double signs, so that they are  $\pm A_0e_0 \pm A_1e_1 \pm A_2e_2 \pm A_3e_3$ . This gives eight cases, namely,

+ + + +	- + + +
+ + + -	+ + - -
+ + - +	+ - - +
+ - + +	+ - + -

The eight apparent cases that would arise by changing all the signs are included in these because the points must be essentially positive. Moreover, no positive point could have three negative signs, because the sum of any three faces of the tetrahedron must be greater than the fourth face. It will be found on trial that six of the bisecting planes will pass through  $\sum(\pm Ae)$  with any one of the above arrangements of sign.

Prob. 21. The twelve points in which the edges of a tetrahedron are cut by the bisecting planes of the opposite dihedral angles fix eight planes, each of which passes through six of them.

Prob. 22. The centroid of the faces of a tetrahedron coincides with the center of the sphere inscribed within the tetrahedron whose vertices are the centroids of the respective faces of the first tetrahedron.

Prob. 23. If any plane be passed through the middle points of two opposite edges of a tetrahedron, it will divide the volume of the tetrahedron into two equal parts.

## ART. 11. THE COMPLEMENT IN SOLID SPACE.

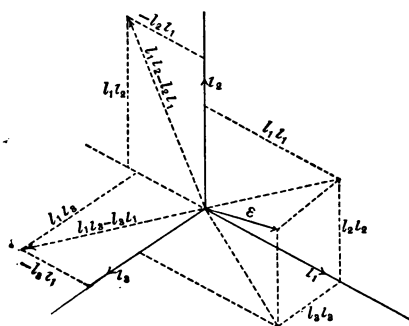
According to the definitions of Art. 8 the complementary relations in a unit normal vector system are as follows:

$$\left. \begin{aligned} |i_1 = i_2 i_3, & \quad |i_2 i_3 = |(|i_1) = i_1 \\ |i_2 = i_3 i_1, & \quad |i_3 i_1 = |(|i_2) = i_2 \\ |i_3 = i_1 i_2, & \quad |i_1 i_2 = |(|i_3) = i_3 \end{aligned} \right\}. \quad (101)$$

Let  $\epsilon = \sum_1^3 l_i$ ; then

$$|\epsilon = l_1 i_1 + l_2 i_2 + l_3 i_3 = \frac{1}{l_1} (l_1 i_1 - l_2 i_2) (l_1 i_1 - l_2 i_2), \quad (102)$$

so that  $|\epsilon$  is a plane-vector. The figure, which is drawn in



isometric projection, shows that the two vectors  $l_1 i_1 - l_2 i_2$  and  $l_1 i_1 - l_3 i_3$ , whose product is  $l_1 \cdot |\epsilon$ , are both perpendicular to  $\epsilon$ ; for the first is perpendicular to  $l_1 i_1 + l_2 i_2$ , which is the orthogonal projection of  $\epsilon$  upon  $i_1 i_2$ , and to  $i_3$ , and therefore is also perpendicular to  $\epsilon$ , while the

second is perpendicular to  $l_1 i_1 + l_3 i_3$ , and to  $i_2$ , and therefore to  $\epsilon$ . Hence  $|\epsilon$  is a plane-vector perpendicular to  $\epsilon$ ; and, since  $|(|\epsilon) = \epsilon$ , the converse is also true, i.e. the complement of a plane-vector is a line-vector normal to it.

The figure shows that  $\epsilon$  is equal to the vector diagonal of the rectangular parallelepiped whose edges have the lengths  $l_1, l_2, l_3$ , hence

$$T\epsilon = \sqrt{l_1^2 + l_2^2 + l_3^2}. \quad (103)$$

Multiply equation (102) by  $\epsilon$ ; therefore

$$\begin{aligned} \epsilon|\epsilon &= (l_1 i_1 + l_2 i_2 + l_3 i_3)(l_1 i_1 + l_2 i_2 + l_3 i_3) \\ &= l_1^2 + l_2^2 + l_3^2 = T^2 \epsilon = \epsilon^2, \end{aligned} \quad (104)$$

so that the co-square of a vector is equal to the square of its tensor. The product  $\epsilon|\epsilon$  is that of a vector  $\epsilon$  into a plane-vector perpendicular to it, as has just been shown; it is there-

fore a volume which is equivalent to  $T\epsilon \cdot T|\epsilon$ ; hence, by (104),  $\epsilon|\epsilon = T\epsilon \cdot T|\epsilon = T^2\epsilon$ , or  $T\epsilon = T|\epsilon$ . Hence, the complement of a vector in solid space is a plane-vector perpendicular to it and having the same tensor, or numerical measure of magnitude.\*

Let a second vector be  $\epsilon' = \sum_1^3 m_i$ ; then

$$\epsilon|\epsilon' = l_1 m_1 + l_2 m_2 + l_3 m_3 = \epsilon'|\epsilon. \quad (105)$$

Now  $\epsilon|\epsilon'$ , being the product of  $\epsilon$  into the plane-vector  $|\epsilon'$ , is the volume of the parallelepiped in the figure, that is,  $T\epsilon T\epsilon' \sin(\text{angle between } \epsilon \text{ and } |\epsilon') = T\epsilon T\epsilon' \cos \epsilon'_1$ . Hence

$$\epsilon|\epsilon' = \epsilon'|\epsilon = l_1 m_1 + l_2 m_2 + l_3 m_3 = T\epsilon T\epsilon' \cos \epsilon'_1. \quad (106)$$

If  $T\epsilon = T\epsilon' = 1$ ,  $l_1 \dots l_3$ ,  $m_1 \dots m_3$  are direction cosines, and (105) gives a proof of the formula for the cosine of the angle between two lines in terms of the direction cosines of the lines. We have also in this case

$\epsilon\epsilon' = (l_1 m_1 - l_2 m_2)|_1 + (l_2 m_3 - l_3 m_2)|_2 + (l_3 m_1 - l_1 m_3)|_3$ , and, taking the co-square,

$$(\epsilon\epsilon')^2 = (\sin \epsilon'_1)^2 = (l_1 m_3 - l_3 m_1)^2 + (l_2 m_3 - l_3 m_2)^2 + (l_3 m_1 - l_1 m_3)^2. \quad (107)$$

$$\text{If} \quad \epsilon|\epsilon' = 0, \quad (108)$$

$\epsilon$  is parallel to the plane-vector perpendicular to  $\epsilon'$ , that is,  $\epsilon$  is perpendicular to  $\epsilon'$ , as is also shown by (106).

Let  $\eta = |\epsilon$ ,  $\eta' = |\epsilon'$ ; then

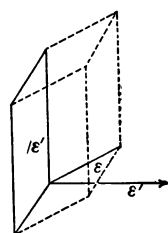
$$\eta|\eta' = |\epsilon \cdot \epsilon' = \epsilon'|\epsilon = \epsilon|\epsilon' = T\epsilon T\epsilon' \cos \epsilon'_1 = T\eta T\eta' \cos \eta'_1. \quad (109)$$

$$\text{and} \quad \eta|\eta' = 0 \quad (110)$$

is the condition of perpendicularity of two plane-vectors. Also either

$$\epsilon|\eta' = 0, \quad \text{or} \quad \eta'|\epsilon = 0, \quad (111)$$

is the condition that a vector shall be perpendicular to a plane-vector, for the first means that  $\epsilon$  is parallel to a vector which is



\* Grassmann (1862), Art. 335.

perpendicular to  $\eta'$ , and the second that  $\eta'$  is parallel to a plane-vector which is perpendicular to  $\epsilon$ .

Equations (71)–(77) of Art. 9 become stereometric vector formulæ if  $\epsilon_1, \epsilon_2$ , etc., be substituted for  $p_1, p_2$ , etc., and  $\eta_1, \eta_2$ , etc., for  $L_1, L_2$ , etc. For instance, (76) gives the vector formulas

$$\epsilon_1 \epsilon_2 | \epsilon'_1 \epsilon'_2 = \begin{vmatrix} \epsilon_1 | \epsilon'_1 & \epsilon_1 | \epsilon'_2 \\ \epsilon_2 | \epsilon'_1 & \epsilon_2 | \epsilon'_2 \end{vmatrix}, \quad \eta_1 \eta_2 | \eta'_1 \eta'_2 = \begin{vmatrix} \eta_1 | \eta'_1 & \eta_1 | \eta'_2 \\ \eta_2 | \eta'_1 & \eta_2 | \eta'_2 \end{vmatrix}. \quad (112)$$

For lack of space no treatment of the complement in a point system in solid space is given.

Exercise 17.—To prove the formulas of spherical trigonometry  $\cos a = \cos b \cos c + \sin b \sin c \cos A$ , and

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}.$$

Take three unit vectors  $\epsilon_1, \epsilon_2, \epsilon_3$  parallel to the radii to the vertices of the spherical triangle, then  $a$ =(angle bet.  $\epsilon_2$  and  $\epsilon_3$ ),  $A$ =(angle bet.  $\epsilon_1 \epsilon_2$  and  $\epsilon_1 \epsilon_3$ ), etc. In eq. (112) put  $\epsilon_1 \epsilon_2$  for  $\epsilon'_1 \epsilon'_2$ ;

$$\begin{aligned} \text{hence } \epsilon_1 \epsilon_2 | \epsilon_1 \epsilon_3 &= \sin b \sin c \cos A = \epsilon_2 | \epsilon_3 - \epsilon_1 | \epsilon_2 \cdot \epsilon_1 | \epsilon_3 \\ &= \cos a - \cos b \cos c. \end{aligned}$$

Again,

$$T(\epsilon_1 \epsilon_2 \cdot \epsilon_1 \epsilon_3) = T(\epsilon_1 \epsilon_2 \epsilon_3 \cdot \epsilon_1) = T\epsilon_1 \epsilon_2 \epsilon_3 = T(\epsilon_2 \epsilon_3 \cdot \epsilon_1 \epsilon_2) = T(\epsilon_1 \epsilon_2 \cdot \epsilon_1 \epsilon_3);$$

$$\text{or } \sin b \sin c \sin A = \sin a \sin c \sin B = \sin a \sin b \sin C,$$

whence we have the second result by dividing by  $\sin a \sin b \sin c$ .

Exercise 18.—Show that in a spherical triangle taken as in Exercise 17,  $\cos \frac{A}{2} = \frac{U\epsilon_1 \epsilon_2 | (U\epsilon_1 \epsilon_2 + U\epsilon_1 \epsilon_3)}{T(U\epsilon_1 \epsilon_2 + U\epsilon_1 \epsilon_3)}$ , whence derive

$$\text{the ordinary value } \sqrt{\frac{\sin s \sin (s-a)}{\sin b \sin c}}.$$

Expanding, the numerator becomes  $1 + U\epsilon_1 \epsilon_2 | U\epsilon_1 \epsilon_3$ , and the denominator  $\sqrt{2(1 + U\epsilon_1 \epsilon_2 | U\epsilon_1 \epsilon_3)}$ . Also there is obtained

$$U\epsilon_1 \epsilon_2 | U\epsilon_1 \epsilon_3 = \frac{\epsilon_1 \epsilon_2 | \epsilon_1 \epsilon_3}{T\epsilon_1 \epsilon_2 T\epsilon_1 \epsilon_3}.$$

The remainder is left to the student.

Prob. 24. If  $\epsilon_1, \epsilon_2, \epsilon_3$ , drawn outward from a point, are taken as three edges of a tetrahedron, show that the six planes perpen-

dicular to the edges at their middle points all pass through the end of the vector  $\rho = \frac{1}{2\epsilon_1\epsilon_2\epsilon_3}(|\epsilon_1\epsilon_2 \cdot \epsilon_1|^2 + |\epsilon_1\epsilon_2 \cdot \epsilon_2|^2 + |\epsilon_1\epsilon_2 \cdot \epsilon_3|^2)$ . (Suggestion. We must have  $(\rho - \frac{1}{2}\epsilon_1)|\epsilon_1 = 0$ , with two other similar expressions.)

Prob. 25. Show that  $\epsilon, |\epsilon\epsilon'$  and  $\epsilon\epsilon'|\epsilon$  are three mutually perpendicular vectors, no matter what the directions of  $\epsilon$  and  $\epsilon'$  may be.

Prob. 26. Let  $\epsilon_1, \epsilon_2, \epsilon_3$  be taken as in Prob. 24; let  $A_0$  be the area of the face of the tetrahedron formed by joining the ends of these vectors, and  $2A_1 = T\epsilon_2\epsilon_3$ , etc.; also  $\theta_1 = \text{Angle between } \epsilon_1\epsilon_2 \text{ and } \epsilon_1\epsilon_3$ , etc.: then show that we have the relation, analogous to that of Prob. 15, Art. 8,

$A_0^2 = A_1^2 + A_2^2 + A_3^2 - 2A_1A_2 \cos \theta_1 - 2A_2A_3 \cos \theta_2 - 2A_1A_3 \cos \theta_3$ . If  $\theta_1 \dots \theta_3$  are right angles, this becomes the space-analog of the proposition regarding the hypotenuse and sides of a right-angled triangle. (Suggestion.  $2A_0 = T(\epsilon_2 - \epsilon_1)(\epsilon_3 - \epsilon_1)$ .)

Prob. 27. There are given three non-coplanar lines  $e_0\epsilon_1, e_0\epsilon_2, e_0\epsilon_3$ ; planes cut these lines at right angles, the sum of the squares of their distances from  $e_0$  being constant. Show that the locus of the common point of these three planes is  $(\rho|\epsilon_1|^2 + (\rho|\epsilon_2|^2 + (\rho|\epsilon_3|^2 = c^2$ , if  $T\epsilon_1 = T\epsilon_2 = T\epsilon_3 = 1$ .

## ART. 12. ADDITION OF SECTS IN SOLID SPACE.

Two lines in solid space will not in general intersect, so that their sum will not be, as in eq. (43), a definite line. For let  $p_1\epsilon_1$  and  $p_2\epsilon_2$  be any two sects: then

$$\begin{aligned} p_1\epsilon_1 + p_2\epsilon_2 &= p_1\epsilon_1 + p_2\epsilon_2 + e_0(\epsilon_1 + \epsilon_2) - e_0(\epsilon_1 + \epsilon_2) \\ &= e_0(\epsilon_1 + \epsilon_2) + (p_1 - e_0)\epsilon_1 + (p_2 - e_0)\epsilon_2; \end{aligned}$$

that is, the sum is a sect passing through an arbitrary point  $e_0$ , and a plane-vector, the sum of the two in the equation. The sum cannot be a single sect unless the two are coplanar; for let  $p_2 = p_1 + x\epsilon_1 + y\epsilon_2 + z\epsilon_3$ ,  $\epsilon_3$  being a vector not parallel to  $\epsilon_1\epsilon_2$ ;

$$\begin{aligned} \text{hence } p_1\epsilon_1 + p_2\epsilon_2 &= p_1\epsilon_1 + (p_1 + x\epsilon_1 + y\epsilon_2 + z\epsilon_3)\epsilon_2 \\ &= p_1(\epsilon_1 + \epsilon_2) + x\epsilon_1(\epsilon_1 + \epsilon_2) + y\epsilon_2\epsilon_2 \\ &= (p_1 + x\epsilon_1)(\epsilon_1 + \epsilon_2) + z\epsilon_2\epsilon_3; \end{aligned}$$

and this cannot reduce to a single sect unless  $z = 0$ , that is, unless  $p_1\epsilon_1$  and  $p_2\epsilon_2$  are coplanar. Since a plane-vector is a line at



$\infty$ , the sum of two lines may always be presented as the sum of a finite line and a line at  $\infty$ .

If the sum of any two sects is equal to the sum of any other two, their products will also be equal, that is, the two pairs will determine tetrahedra of equal volumes. For let  $L_1 + L_2 = L_3 + L_4$ ; then squaring we have  $L_1 L_2 = L_3 L_4$ , since  $L_1 L_4 = 0$ , etc.

An infinite number of pairs of sects can be found such that the sum of each pair is equal to the sum of any given pair; for let a given pair be  $p_1 \epsilon_1 + p_2 \epsilon_2$ , and take a new pair

$$\begin{aligned} & (x_1 p_1 + x_2 p_2)(u_1 \epsilon_1 + u_2 \epsilon_2) + (y_1 p_1 + y_2 p_2)(v_1 \epsilon_1 + v_2 \epsilon_2) \\ &= (x_1 u_1 + y_1 v_1) p_1 \epsilon_1 + (x_2 u_2 + y_2 v_2) p_2 \epsilon_2 + \\ & \quad (x_1 u_2 + y_1 v_2) p_1 \epsilon_2 + (x_2 u_1 + y_2 v_1) p_2 \epsilon_1. \end{aligned}$$

This will be equal to the given pair if we have

$$x_1 u_1 + y_1 v_1 = x_2 u_2 + y_2 v_2 = 1, \text{ and } x_1 u_2 + y_1 v_2 = x_2 u_1 + y_2 v_1 = 0.$$

Since there are eight arbitrary quantities with only four equations of condition, the desired result can evidently be accomplished in an infinite number of ways.

Let  $p_1 \epsilon_1, p_2 \epsilon_2, \dots, p_n \epsilon_n$  be  $n$  sects, and let  $S$  be their sum, and  $e_0$  any point, then

$$S = \sum_1^n p_i \epsilon_i \equiv e_0 \sum \epsilon_i - e_0 \sum \epsilon_i + \sum p_i \epsilon_i = e_0 \sum \epsilon_i + \sum (p_i - e_0) \epsilon_i, \dots \quad (113)$$

the sum of a sect and a plane-vector as before.

If  $\sum (p_i - e_0) \epsilon_i$  is parallel to  $\sum \epsilon_i$  it may be written as the product of some vector  $\epsilon'$  into  $\sum \epsilon_i$ , that is,  $\epsilon' \sum \epsilon_i$ , when the sum becomes  $S = e_0 \sum \epsilon_i + \epsilon' \sum \epsilon_i = (e_0 + \epsilon') \sum \epsilon_i$ , a sect, because  $e_0 + \epsilon'$  is a point. In no other case does  $S$  reduce to a single sect. If  $\sum \epsilon_i = 0$ ,  $S$  becomes a plane-vector. Of the two parts composing  $S$ , the sect will be unchanged in magnitude and direction if  $e_0$  be moved to a new position, while the plane-vector will in general be altered. It is proposed to show that a point  $q$  may be substituted for  $e_0$  such that the plane-vector will be perpendicular to  $\sum \epsilon_i$ . Writing

$$S \equiv q \sum \epsilon_i - (q - e_0) \sum \epsilon_i + \sum (p_i - e_0) \epsilon_i,$$

and, for brevity, putting  $q - e_0 = \rho$ ,  $\sum \epsilon_i = \alpha$ ,  $\sum (p_i - e_0) \epsilon_i = |\beta$ , so that

$$S \equiv q \alpha - \rho \alpha + |\beta, \quad (114)$$

we must have for perpendicularity, by (111),

$$\begin{aligned} (|\beta - \rho\alpha|)\alpha = 0 &= |\beta\alpha - \rho\alpha|\alpha, \\ \text{or } \rho\alpha|\alpha &\equiv \alpha.\rho|\alpha - \rho.\alpha^2 = |\beta\alpha. \end{aligned} \quad (115)$$

The second member is obtained from the first by substituting in eq. (74)  $\rho$  for  $p$ , and  $\alpha$  for  $p$ , and  $q$ , in accordance with the statement at the end of Art. 11. If in (115) we make  $\rho|\alpha = 0$ ,  $\rho$  will be the vector from  $e_0$  to  $q$  taken perpendicularly to  $\alpha$ , say

$$\rho_1 = |\alpha\beta \div \alpha^2 = q_1 - e_0. \quad (116)$$

Since  $\alpha$  and  $\beta$  are known, the required point has been found. Multiply (115) by  $\alpha$ ; then, using (75),

$$-\alpha\rho.\alpha^2 \equiv \rho\alpha.\alpha^2 = \alpha|\beta\alpha = |\beta.\alpha^2 - |\alpha.\alpha|\beta,$$

whence, substituting in (114),

$$S = q\alpha + \frac{\alpha|\beta}{\alpha^2}.\alpha = q\Sigma\epsilon + \frac{\Sigma\epsilon\Sigma(p - e_0)\epsilon}{(\Sigma\epsilon)^2}.\Sigma\epsilon. \quad (117)$$

This may be called the normal form of  $S$ .\*

The sects of this article represent completely the geometric properties of forces, hence all that has been shown applies immediately to a system of forces in solid space. We have only to substitute the words force and couple for sect and plane-vector. The resultant action of any system of forces is  $S$ , called by Ball in his Theory of Screws "a wrench." The condition for equilibrium is  $S = 0$ , which gives at once

$$\Sigma\epsilon = 0 \quad \text{and} \quad \Sigma(p - e_0)\epsilon = 0; \quad (118)$$

since otherwise we must have  $e_0\Sigma\epsilon = -\Sigma(p - e_0)\epsilon$ , which is an impossibility. The line  $q\Sigma\epsilon$  is the central axis of the system of forces  $S$ .

Lack of space forbids a further development of the subject, but what has been given in this article will indicate the perfect adaptability of this method to the requirements of mechanics.

Exercise 19.—Reduce  $p_1\epsilon_1 + p_2\epsilon_2 = S$  to its normal form.  $S \equiv e_0(\epsilon_1 + \epsilon_2) + (p_1 - e_0)\epsilon_1 + (p_2 - e_0)\epsilon_2$ . For convenience suppose  $p_1$  and  $p_2$  to be taken at the ends of the common per-

\* Grassmann (1862), Art. 346.

pendicular on  $p_1\epsilon_1$  and  $p_2\epsilon_2$ , and moreover let  $e_0 = \frac{1}{2}(p_1 + p_2)$ ,  $p_1 - e_0 = \iota = -(p_2 - e_0)$ ; then  $\iota|\epsilon_1 = \iota|\epsilon_2 = 0$ . Accordingly

$$S \equiv e_0(\epsilon_1 + \epsilon_2) + \iota(\epsilon_1 - \epsilon_2) = q(\epsilon_1 + \epsilon_2) + \frac{(\epsilon_1 + \epsilon_2)\iota(\epsilon_1 - \epsilon_2)}{(\epsilon_1 + \epsilon_2)^2} \cdot |(\epsilon_1 + \epsilon_2)$$

$$= q(\epsilon_1 + \epsilon_2) + \frac{2\iota\epsilon_1\epsilon_2}{(\epsilon_1 + \epsilon_2)^2} \cdot |(\epsilon_1 + \epsilon_2).$$

$$\text{By (II 6), } q - e_0 = -\frac{|\beta \cdot \alpha|}{\alpha^2} = -\frac{\iota(\epsilon_1 - \epsilon_2) \cdot |(\epsilon_1 + \epsilon_2)|}{(\epsilon_1 + \epsilon_2)^2}$$

$$= \frac{\iota \cdot (\epsilon_1 - \epsilon_2) |(\epsilon_1 + \epsilon_2)|}{(\epsilon_1 + \epsilon_2)^2}, \text{ by (74), } = \frac{\epsilon_1^2 - \epsilon_2^2}{(\epsilon_1 + \epsilon_2)^2} \cdot \iota.$$

Hence the normal form of  $S$  is

$$S = \left( e_0 + \frac{\epsilon_1^2 - \epsilon_2^2}{(\epsilon_1 + \epsilon_2)^2} \cdot \iota \right) (\epsilon_1 + \epsilon_2) + \frac{2\iota\epsilon_1\epsilon_2}{(\epsilon_1 + \epsilon_2)^2} \cdot |(\epsilon_1 + \epsilon_2).$$

Exercise 20.—Forces are represented by the six edges of a tetrahedron  $e_0e_1, e_0e_2, e_0e_3, e_1e_2, e_1e_3, e_2e_3$ ; find the  $S$ , reduce to normal form, and consider the special case when three dihedral angles are right angles.  $S \equiv e_0(e_1 + e_2 + e_3) + e_1e_2 + e_2e_3 + e_1e_3$   
 $\equiv e_0(\epsilon_1 + \epsilon_2 + \epsilon_3) + (e_2 - e_1)(e_3 - e_1) \equiv e_0(\epsilon_1 + \epsilon_2 + \epsilon_3) + (\epsilon_2 - \epsilon_1)(\epsilon_3 - \epsilon_1)$   
 $\equiv e_0(\epsilon_1 + \epsilon_2 + \epsilon_3) + \epsilon_2\epsilon_3 + \epsilon_3\epsilon_1 + \epsilon_1\epsilon_2$ , in which  $\epsilon_1 = e_1 - e_0$ , etc. Hence

$$S \equiv \left( e_0 + \frac{(\epsilon_2\epsilon_3 + \epsilon_3\epsilon_1 + \epsilon_1\epsilon_2) |(\epsilon_1 + \epsilon_2 + \epsilon_3)|}{(\epsilon_1 + \epsilon_2 + \epsilon_3)^2} \right) (\epsilon_1 + \epsilon_2 + \epsilon_3)$$

$$+ \frac{3\epsilon_1\epsilon_2\epsilon_3}{(\epsilon_1 + \epsilon_2 + \epsilon_3)^2} \cdot |(\epsilon_1 + \epsilon_2 + \epsilon_3).$$

For the rectangular tetrahedron let  $\epsilon_1 = a\iota_1$ ,  $\epsilon_2 = b\iota_2$ ,  $\epsilon_3 = c\iota_3$ ,  $\iota_1, \iota_2, \iota_3$  being unit normal vectors. Then we find

$$S \equiv \left( e_0 + \frac{a(c^2 - b^2)\iota_1 + b(a^2 - c^2)\iota_2 + c(b^2 - a^2)\iota_3}{a^2 + b^2 + c^2} \right) (a\iota_1 + b\iota_2 + c\iota_3)$$

$$+ \frac{3abc}{a^2 + b^2 + c^2} \cdot |a\iota_1 + b\iota_2 + c\iota_3.$$

Exercise 21.—A pole 50 feet high stands on the ground and is held erect by three guy-ropes symmetrically arranged about it, attached to its top and to pegs in the ground 50 feet from the pole. The wind blows against the pole with a pressure of 50 pounds in the direction  $e_0 - p$ , when  $e_0$  is at the bottom of

the pole, and  $p$  divides the distance between two of the pegs in the ratio  $\frac{m}{n}$ : find the tension on the guys and the pressure on the ground.

Evidently only two of the guys will be in tension; let their pegs be at  $e_1$  and  $e_2$ , and let  $e_3$  be at the top of the pole, and  $w$  the weight of the pole. Then  $p = \frac{me_1 + ne_2}{m + n}$ , and the equation of equilibrium is

$$50 \cdot \frac{(e_0 + e_3)(e_0 - p)}{2T(e_0 - p)} + \frac{25e_0(p - e_0)}{T(e_0 - p)} + \frac{(x + w)e_0e_1}{Te_0e_1} + \frac{ye_2e_1}{Te_2e_1} + \frac{ze_2e_3}{Te_2e_3} = 0.$$

$$Te_0e_2 = 50, Te_2e_1 = Te_2e_3 = 50\sqrt{2}, T(p - e_0) = T\left(\frac{me_1 + ne_2}{m + n} - e_0\right) \\ = T\left(\frac{m(e_1 - e_0) + n(e_2 - e_0)}{m + n}\right) = \frac{50}{m + n}T(m\epsilon_1 + n\epsilon_2), \text{ if } \epsilon_1 = U(e_1 - e_0)$$

$$\text{and } \epsilon_2 = U(e_2 - e_0); \text{ then } T(p - e_0) = \frac{50}{m + n}\sqrt{m^2 + n^2 - mn},$$

because  $\epsilon_1^2 = \epsilon_2^2 = 1$ , and  $\epsilon_1|\epsilon_2 = \cos 120^\circ = -\frac{1}{2}$ . Hence the equation of equilibrium becomes

$$\frac{25e_2((m + n)e_0 - me_1 - ne_2)}{\sqrt{m^2 + n^2 - mn}} + (x + w)e_0e_1 + \frac{y}{\sqrt{2}}e_2e_1 + \frac{z}{\sqrt{2}}e_2e_3 = 0.$$

Multiply successively by  $e_1e_1$ ,  $e_2e_2$ , and  $e_3e_3$ , and we obtain

$$\frac{x + w}{m + n} = \frac{y}{m\sqrt{2}} = \frac{z}{n\sqrt{2}} = \frac{25}{\sqrt{m^2 + n^2 - mn}},$$

$y$  and  $z$  being the tensions, and  $x + w$  the upward pressure.

Prob. 28. Three equal poles are set up so as to form a tripod, and are mutually perpendicular; a weight  $w$  hangs upon a rope which passes over a pulley at the top of the tripod, and thence down under a pulley at the ground at a point  $p = \sum_1 l$ , in which  $e_1 \dots e_3$  are at the feet of the poles, and  $\sum_1 l = 1$ ; if the rope is pulled

so as to raise  $w$ , show that the pressures on the poles, supposing the pulleys frictionless, are

$$w\left(\frac{l_1}{\sqrt{\sum l^2}} + \frac{1}{\sqrt{3}}\right), \quad w\left(\frac{l_2}{\sqrt{\sum l^2}} + \frac{1}{\sqrt{3}}\right), \quad w\left(\frac{l_3}{\sqrt{\sum l^2}} + \frac{1}{\sqrt{3}}\right).$$

Prob. 29. Six equal forces act along six successive edges of a cube which do not meet a given diagonal; show that if the edges of the cube be parallel to  $i_1, i_2, i_3$ , and  $F$  be the magnitude of each force, then  $S = -2F|(i_1 + i_2 + i_3)$ , if the diagonal taken be parallel to  $i_1 + i_2 + i_3$ .

Prob. 30. Three forces whose magnitudes are 1, 2, and 3 act along three successive non-coplanar edges of a cube; show that the normal form of  $S$  is

$$S = (e_0 + \frac{1}{2}i_1 + \frac{1}{2}i_2 - \frac{1}{2}i_3)(i_1 + 2i_2 + 3i_3) + \frac{1}{4}|(i_1 + 2i_2 + 3i_3).$$

Prob. 31. Forces act at the centroids of the faces of a tetrahedron, perpendicular and proportional to the faces on which they act, and all directed inwards, or else all outwards; show that they are in equilibrium.

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